HIGHEST WEIGHT VECTORS AND TRANSMUTATION

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SUMMARY. Let $G = \mathrm{GL}_n$ be the general linear group over an algebraically closed field k, let $\mathfrak{g} = \mathfrak{gl}_n$ be its Lie algebra and let U be the subgroup of G which consists of the upper uni-triangular matrices. Let $k[\mathfrak{g}]$ be the algebra of polynomial functions on $\mathfrak{g}$ and let $k[\mathfrak{g}]^G$ be the algebra of invariants under the conjugation action of G. In characteristic zero, we give for all dominant weights $\chi \in \mathbb{Z}^n$ finite homogeneous spanning sets for the $k[\mathfrak{g}]^G$-modules $k[\mathfrak{g}]^U_\chi$ of highest weight vectors. This result (with some mistakes) was already given without proof by J. F. Donin. Then we do the same for tuples of $n \times n$-matrices under the diagonal conjugation action. Furthermore we extend our earlier results in positive characteristic and give a general result which reduces the problem to giving spanning sets of the highest weight vectors for the action of $\mathrm{GL}_r \times \mathrm{GL}_s$ on tuples of $r \times s$ matrices. This requires the technique called "transmutation" by R. Brylinsky which is based on an instance of Howe duality. In the cases that $\chi_n \geq -1$ or $\chi_1 \leq 1$ this leads to new spanning sets for the modules $k[\mathfrak{g}]^U_\chi$.

INTRODUCTION

Let k be an algebraically closed field and let GL_n be the group of invertible $n \times n$ matrices with entries in k and let T_n and U_n be the subgroups of diagonal matrices and of upper uni-triangular matrices. The group GL_n acts on the k-vector space Mat_n of $n \times n$ matrices with entries in k via $S \cdot A = SAS^{-1}$ and therefore on its coordinate ring $k[\mathrm{Mat}_n]$ via $(S \cdot f)(A) = f(S^{-1}AS)$. We identify the character group of T_n with $\mathbb{Z}^n$: if $\chi \in \mathbb{Z}^n$, then $D \mapsto \prod_{i=1}^n D_{ii}^{\chi_i}$ is the corresponding character of T_n. We will call the characters of T_n *weights* of T_n or GL_n and the weights χ of T_n for which the corresponding weight space M_χ of a given T_n module M is nonzero will be called *weights* of M. We say that $\chi \in \mathbb{Z}^n$ is *dominant* if it is weakly decreasing.

We will be interested in finding finite homogeneous spanning sets for the $k[\mathrm{Mat}_n]^{\mathrm{GL}_n}$-modules $k[\mathrm{Mat}_n]^{U_n}_\chi$ of highest weight vectors. As is well-known, such a module is nonzero if and only if χ is dominant and has coordinate sum zero. A weight $\chi \in \mathbb{Z}^n$ with this property can uniquely be written as $\chi = [\lambda, \mu] := (\lambda_1, \lambda_2, \ldots, 0, \ldots, 0, \ldots, -\mu_2, -\mu_1)$ were λ and μ are partitions with $|\lambda| = |\mu|$ and $l(\lambda) + l(\mu) \leq n$. Here $l(\lambda)$ denotes the length of a partition λ and $|\lambda|$ denotes its coordinate sum. As usual partitions are extended with zeros if necessary.

The nilpotent cone $\mathcal{N}_n = \{A \in \mathrm{Mat}_n \mid A^n = 0\}$ is a GL_n-stable closed subvariety of Mat_n. Using the graded Nakayama Lemma it is easy to see that it suffices to find finite homogeneous spanning sets for the vector spaces of highest

weight vectors $k[\mathcal{N}_n]^{U_n}_\chi$ in the coordinate ring of $\mathcal{N}_n$. For background on the adjoint action on $k[\mathrm{Mat}_n]$ and $k[\mathcal{N}_n]$, e.g. graded character formulas, we refer to the introduction of [24] and the references in there.

In [5] a process called *transmutation* is applied to understand the conjugation action of GL_n on the nilpotent cone. We briefly explain the idea and for simplicity we assume that k has characteristic 0. Let G, H be reductive groups and let Y be an affine $G \times H$-variety such that $k[Y] = \bigoplus_{i \in I} L_i^* \otimes M_i$ where the L_i are mutually nonisomorphic G-modules and the M_i are mutually nonisomorphic H-modules. Then Y can be used as a "catalyst" for transmutation as follows. If V is an affine G-variety, then the $W = Y \times^G V := (Y \times V)//G$ is an H variety, the H-irreducibles that show up in $k[W]$ are the M_i, and the multiplicity of M_i in $k[W]$ is the same as that of L_i in $k[V]$. The goal is to find for a given V a suitable H and Y for which the resulting W is much simpler than V, but still contains enough interesting information coming from V. In [5] R. Brylinsky applied this technique to the closed GL_n-stable subvariety $V = \mathcal{N}_{n,m} = \{A \in \mathcal{N}_n \mid A^{m+1} = 0\}$ of Mat_n and $G = \mathrm{GL}_n$. She showed that in this case for $H = \mathrm{GL}_r \times \mathrm{GL}_s$ and a suitable catalyst Y the transmuted variety W is a certain closed subvariety of Mat^m_{rs} which is all of Mat^m_{rs} if n is sufficiently big relative to m, r and s. Here $\mathrm{GL}_r \times \mathrm{GL}_s$ acts on Mat^m_{rs} via $((R, S) \cdot \underline{A})_i = R A_i S^{-1}$, $\underline{A} = (A_1, \ldots, A_m) \in \mathrm{Mat}^m_{rs}$, and on the coordinate ring $k[\mathrm{Mat}^m_{rs}]$ via $((R, S) \cdot f)(\underline{A}) = f((R^{-1}, S^{-1}) \cdot \underline{A})$. The correspondence between the irreducibles for the two groups is in terms of the labels given by $\chi = [\lambda, \mu] \leftrightarrow (-\mu^{\mathrm{rev}}, \lambda)$, where μ^{rev} is the reversed r-tuple of μ.

In this paper we give finite homogeneous spanning sets for the vector spaces $k[\mathcal{N}_n]^{U_n}_\chi$ in characteristic 0 using transmutation and some results of J. Donin on skew representations for the symmetric group. Donin already stated our result Theorem 4 in [7] and [8], but gave no proof.[1] By making Brylinsky's results explicit in terms of highest weight vectors we can prove the result of Donin on the highest weight vectors in $k[\mathrm{Mat}_n]$. It turns out that the method of "transmutation" works in any characteristic and for certain special weights we can give bases for the highest weight vectors in the coordinate ring of the transmuted variety which then give spanning sets for the highest weight vectors in $\mathcal{N}_n$.

The paper is organised as follows. In Section 1 we introduce some notation, e.g. for diagrams and tableaux, and we state some well-known results from the literature on the algebra $k[\mathrm{Mat}_n]^{\mathrm{GL}_n}$, reduction to the nilpotent cone and good filtrations that we will need. In Section 2 we show in Theorem 1 that the technique of transmutation works in our case in any characteristic. Our main tool here is Donkin's results on good pairs of varieties [9]. We can apply Theorem 1 in arbitrary characteristic for weights χ with $\chi_n \geq -1$ or $\chi_1 \leq 1$. For the corresponding $\mathrm{GL}_r \times \mathrm{GL}_s$-weights we give in Theorem 2 bases for the spaces of highest weight vectors in the coordinate ring of the "transmuted space" Mat^m_{rs}.

In Section 3 we always assume that our field k has characteristic 0. In Section 3.1 we first develop the necessary results on skew representations of the

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[1]The "proof" of [7, Prop. 4.1] is not convincing.

symmetric group. What we need is explicit polytabloid bases for the "coinvariants" for a Young subgroup in a Specht module, see Proposition 3. Although Donin didn't prove Theorem 4 or its analogue for the nilpotent cone, he did give proofs in [7] for most of the required results on the symmetric group. However, his proofs are mostly incomplete and sometimes incorrect. So we give an account with complete proofs. In Section 3.2 we give bases for the spaces of highest weight vectors in the coordinate ring of the "transmuted space" Mat_{rs}^m. Combined with Theorem 1 this gives finite homogeneous spanning sets for the vector spaces $k[\mathcal{N}_n]_\chi^{U_n}$ in characteristic 0. This can then further be combined with Lemma 1 below to obtain finite homogeneous spanning sets for the $k[\mathrm{Mat}_n]^{\mathrm{GL}_n}$-modules $k[\mathrm{Mat}_n]_\chi^{U_n}$. In Section 3.3 we briefly describe a generalisation to several matrices and how to obtain spanning sets for the $k[\mathrm{Mat}_n^l]^{\mathrm{GL}_n}$-modules $k[\mathrm{Mat}_n^l]_\chi^{U_n}$.

1. Preliminaries

Throughout this paper k is an algebraically closed field of arbitrary characteristic. All our varieties are affine. The groups GL_n, T_n, U_n and the actions of GL_n on Mat_n and $\mathcal{N}_{n,m}$ and of $\mathrm{GL}_r \times \mathrm{GL}_s$ on Mat_{rs}^m are as in Section 1.

1.1. **The graded Nakayama Lemma.** As is well-known the algebra $k[\mathrm{Mat}_n]^{\mathrm{GL}_n}$ is generated by the algebraically independent functions $s_1, \ldots, s_n$ given by $s_i(A) = \mathrm{tr}(\wedge^i A)$, where $\wedge^i A$ denotes the i-th exterior power of A. Furthermore, the s_i generate the vanishing ideal of $\mathcal{N}_n$. If m is the dimension of the zero weight space of $\nabla_{\mathrm{GL}_n}(\chi)$, then $k[\mathcal{N}_n]_\chi^{U_n}$ has dimension m and $k[\mathrm{Mat}_n]_\chi^{U_n}$ is a free $k[\mathrm{Mat}_n]^{\mathrm{GL}_n}$-module of rank m. The following lemma is an application of the graded Nakayama Lemma.

Lemma 1. *Let* $f_1, \ldots, f_l \in k[\mathrm{Mat}_n]_\chi^{U_n}$ *be homogeneous. If the restrictions* $f_1|_{\mathcal{N}_n}, \ldots, f_l|_{\mathcal{N}_n}$ *span* $k[\mathcal{N}_n]_\chi^{U_n}$, *then* $f_1, \ldots, f_l$ *span* $k[\mathrm{Mat}_n]_\chi^{U_n}$ *as a* $k[\mathrm{Mat}_n]^{\mathrm{GL}_n}$-*module. The same holds with "span" replaced by "form a basis of".*

We refer to [23, Lem. 2, Prop. 1] for references and explanation.

1.2. **Good filtrations.** For G a reductive group and χ a dominant weight relative to a Borel subgroup $B = TU$ we denote the standard or Weyl module corresponding to χ by $\Delta_G(\chi)$ and the costandard or induced module corresponding to χ by $\nabla_G(\chi)$. We have $\Delta_G(\chi) \cong \nabla_G(-w_0(\chi)^*$, where w_0 is the longest element in the Weyl group. The module $\nabla_G(\chi)$ has simple socle and the module $\Delta_G(\chi)$ has simple top, both isomorphic to the irreducible $L_G(\chi)$ of highest weight χ. In characteristic 0 we have $\Delta_G(\chi) \cong \nabla_G(\chi) \cong L_G(\chi)$. The main property of these modules that we will use is that all dominant χ_1 and χ_2, $\mathrm{Ext}_G^1(\Delta_G(\chi_1), \nabla_G(\chi_2)) = 0$ and $\mathrm{Hom}_G(\Delta_G(\chi_1), \nabla_G(\chi_2)) = k$ if $\chi_1 = \chi_2$ and $\{0\}$ otherwise. See [17, II.4.13]. A G-module M is said to have a *good filtration* if it has a G-module filtration $0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$, $\bigcup_{i \geq 0} M_i = M$, such that each quotient M_i/M_{i-1} is isomorphic to some induced module $\nabla_G(\chi)$. If M has a good filtration, the number of quotients isomorphic to $\nabla_G(\chi)$ is independent of the good filtration and equal to $\dim M_\chi^U$. If k has characteristic 0, then every G-module has a good filtration. For more details we refer to [17,

II.4.16,17]. For example, a direct summand of a module with a good filtration has a good filtration.

1.3. **Graded characters.** If $M = \bigoplus_{i \geq 0} M_i$ is a graded vector space with $\dim M_i < \infty$ for all i, then the *graded dimension* of M is the polynomial $\sum_i \dim M_i z^i$. Here one can use for z any other grading variable. Similarly, if G is a general linear group, $M = \bigoplus_{i \geq 0} M_i$ a graded G-module with a good filtration, and $\nabla_G(\chi)$ has finite good filtration multiplicity in M, then the *graded good filtration multiplicity* of $\nabla_G(\chi)$ in M is the polynomial $\sum_i (M_i : \nabla_G(\chi)) z^i$, where $(M_i : \nabla_G(\chi))$ is the good filtration multiplicity of $\nabla_G(\chi)$ in M_i. Note that by the above the graded good filtration multiplicity of $\nabla_G(\chi)$ in M is the graded dimension of M_χ^U. We say that one graded dimension or multiplicity is $\leq$ another if this is true coefficient-wise.

1.4. **Good pairs.** Recall from [9] that an affine variety V on which a reductive group G acts is called *good* if $k[V]$ has a good filtration. Furthermore, if A is a closed G-stable subvariety of V, then (V, A) is called a *good pair of G-varieties* if the vanishing ideal of A in $k[V]$ has a good filtration. In this case A is itself a good G-variety. If (V, A) is a good pair of G-varieties, then the restriction map $k[V]_\chi^U \to k[A]_\chi^U$ is surjective by [17, II.4.13].

1.5. **Skew Young diagrams and tableaux.** For λ a partition of n we denote the nilpotent orbit which consists of the matrices whose Jordan normal form has block sizes $\lambda_1, \cdots, \lambda_{l(\lambda)}$, by $\mathcal{O}_\lambda$. For λ, μ partitions of n, we say that $\lambda \geq \mu$ if $\sum_{j=1}^i \lambda_j \geq \sum_{j=1}^i \mu_j$ for $i = 1, \ldots, n-1$. This order is called the *dominance order*. In [12, Prop 1.6] it was proved that $\overline{\mathcal{O}}_\lambda \supseteq \mathcal{O}_\mu$ if and only if $\lambda \geq \mu$. Here $\overline{\mathcal{O}}_\lambda$ denotes the closure of the orbit $\mathcal{O}_\lambda$. Since $\mathcal{N}_{n,m-1}$ is the union of the $\mathcal{O}_\lambda$ with $\lambda_1 \leq m$, it follows easily that $\mathcal{N}_{n,m-1} = \overline{\mathcal{O}}_{m^q r}$, where q and r are quotient and remainder under division of n by m.

We will denote the transpose of a partition λ by λ' and we will identify each partition λ with the corresponding Young diagram $\{(i,j) \mid 1 \leq i \leq l(\lambda), 1 \leq j \leq \lambda_i\}$. The $(i,j) \in \lambda$ are called the *boxes* or *cells* of λ. More generally, if λ, μ are partitions with $\lambda \supseteq \mu$, then we denote the diagram λ with the boxes of μ removed by λ/μ and call it the *skew Young diagram* associated to the pair (λ, μ). Of course the skew diagram λ/μ does not determine λ and μ. We denote the number of boxes in a skew diagram E by $|E|$. We define Δ_t to be the diagram
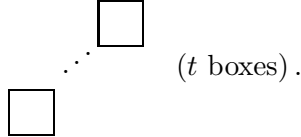
$$(t \text{ boxes}).$$

Let E be as skew diagram with t boxes. A *skew tableau* of shape E is a mapping $T : E \to \mathbb{N} = \{1, 2, \ldots\}$. A skew tableau of shape E is called *row-ordered* if its entries are weakly increasing along rows, *strictly row-ordered* if its entries are strictly increasing along rows, and it is called *ordered* if its entries are weakly increasing along rows and along columns. The notions column-ordered and strictly column-ordered are defined in a completely analogous way.

A skew tableau of shape E is called *semi-standard* if its entries are weakly increasing along the rows and strictly increasing along the columns, and it is called *row semi-standard* if its entries are strictly increasing along the rows and weakly increasing along the columns. It is called a *t-tableau* if its entries are the numbers $1, \ldots, t$ (so the entries must be distinct) and it is called *standard* if it is a t-tableau and its entries are (strictly) increasing along rows and along columns. We will associate to E two special skew tableaux T_E and S_E as follows. We define T_E by filling in the numbers $1, \ldots, t$ row by row from left to right and top to bottom and we define S_E by filling the boxes in the i-th row with i's. So T_E is standard and S_E is semi-standard. Two tableaux S and T of shape E are called *row equivalent* if, for each i, the i-th row of F is a permutation of the i-th row of T. The notion of column equivalence is defined in a completely analogous way. Finally, if m is the biggest integer occurring in a tableau T, or 0 if T is empty, then the *weight* of T is the m-tuple whose i-th component is the number of occurrences of i in T. Sometimes we will also consider the weight of T as an m'-tuple for some $m' \geq m$ by extending it with zeros.

2. TRANSMUTATION AND SEMI-INVARIANTS IN CHARACTERISTIC p

Let r, s be integers ≥ 0 with $r + s \leq n$. We denote the variety of pairs $(A, B) \in \mathrm{Mat}_{rn} \times \mathrm{Mat}_{ns}$ with $AB = 0$ by $Y_{r,s,n}$ and for m an integer ≥ 2 we define the maps $\varphi_{r,s,n,m}$ and $\overline{\varphi}_{r,s,n,m}$ by

$$\varphi_{r,s,n,m} : (A, B, X) \mapsto (AB, AXB, \ldots, AX^m B)$$
$$: \mathrm{Mat}_{rn} \times \mathrm{Mat}_{ns} \times \mathrm{Mat}_n \to \mathrm{Mat}_{rs} \times \mathrm{Mat}_{rs}^m$$
$$\overline{\varphi}_{r,s,n,m} : (A, B, X) \mapsto (s_1(X), \ldots, s_n(X), \varphi_{r,s,n,m}(A, B, X))$$
$$: \mathrm{Mat}_{rn} \times \mathrm{Mat}_{ns} \times \mathrm{Mat}_n \to k^n \times \mathrm{Mat}_{rs} \times \mathrm{Mat}_{rs}^m .$$

We will denote several of the restrictions of these maps by the same symbol. The group $\mathrm{GL}_{r,s,n} := \mathrm{GL}_r \times \mathrm{GL}_s \times \mathrm{GL}_n$ acts on $\mathrm{Mat}_{rn} \times \mathrm{Mat}_{ns}$ via $(S, T, U) \cdot (A, B) = (SAU^{-1}, UBT^{-1})$ and on $\mathrm{Mat}_{rn} \times \mathrm{Mat}_{ns} \times \mathrm{Mat}_n$ via $(S, T, U) \cdot (A, B, X) = (SAU^{-1}, UBT^{-1}, UXU^{-1})$. Note that $Y_{r,s,n}$ is a $\mathrm{GL}_{r,s,n}$-stable closed subvariety of $\mathrm{Mat}_{rn} \times \mathrm{Mat}_{ns}$. Note also that $\varphi_{r,s,n,m}$ and $\overline{\varphi}_{r,s,n,m}$ are equivariant for the action of $\mathrm{GL}_{r,s,n}$ if we let GL_n act trivially on $k^n \times \mathrm{Mat}_{rs} \times \mathrm{Mat}_{rs}^m$ and $\mathrm{GL}_r \times \mathrm{GL}_s$ trivially on k^n and via its obvious diagonal action on $\mathrm{Mat}_{rs} \times \mathrm{Mat}_{rs}^m$.

We consider $\mathrm{Mat}_{rs} \times \mathrm{Mat}_{rs}^m$ as a closed subvariety of $k^n \times \mathrm{Mat}_{rs} \times \mathrm{Mat}_{rs}^m$ by taking the first n scalar components zero and we consider Mat_{rs}^m as a closed subvariety of $\mathrm{Mat}_{rs} \times \mathrm{Mat}_{rs}^m$ by taking the first matrix component the zero matrix. So $\varphi_{r,s,n,m} = \overline{\varphi}_{r,s,n,m}$ on $\mathrm{Mat}_{rn} \times \mathrm{Mat}_{ns} \times \mathcal{N}_n$ and $\varphi_{r,s,n,m}(Y_{r,s,n} \times \mathcal{N}_n) \subseteq \mathrm{Mat}_{rs}^m$. If $l \geq m$, then we consider Mat_{rs}^m as a closed subvariety of Mat_{rs}^l by extending an m-tuple of $r \times s$ matrices with zero matrices to an l-tuple of $r \times s$ matrices. So $\varphi_{r,s,n,l} = \varphi_{r,s,n,m}$ on $\mathrm{Mat}_{rn} \times \mathrm{Mat}_{ns} \times \mathcal{N}_{n,m}$ if $l \geq \min(m, n - 1)$. When r and s are fixed we denote the image $\varphi_{r,s,n,m}(Y_{r,s,n} \times \mathcal{N}_{n,m}) \subseteq \mathrm{Mat}_{rs}^m$ by $W_{n,m}$.

We will use the embedding of Mat_n in $Y_{r,s,n} \times \mathrm{Mat}_n$ which is given by

$$X \mapsto (E_r, F_s, X) ,$$

where $E_r = \begin{bmatrix} 0 & I_r \end{bmatrix} \in \mathrm{Mat}_{rn}$, $F_s = \begin{bmatrix} I_s \\ 0 \end{bmatrix} \in \mathrm{Mat}_{ns}$. Then $\varphi_{r,s,n,m}$ can be restricted to Mat_n and $\varphi_{r,s,n,m}(X)$ consists of the lower left $r \times s$ corners of the first m powers of X.

Any point of $Y_{r,s,n}$ is contained in an irreducible curve which also contains a point $(A, B) \in Y_{r,s,n}$ with A and B of maximal rank r and s (see e.g. [5, p38]) and if (A, B) is such a point, then it is easy to see that $g \cdot (A, B) = (E_r, F_s)$ for some $g \in \mathrm{GL}_n$. It follows that $Y_{r,s,n}$ is irreducible and that $\varphi_{r,s,n,m}(\mathcal{N}_{n,m})$ is dense in $W_{n,m}$.

We will use the $\mathrm{GL}_{r,s,n}$-variety $Y_{r,s,n}$ as the catalyst for the transmutation from GL_n-varieties to $\mathrm{GL}_r \times \mathrm{GL}_s$-varieties. We will mainly be interested in applying this transmutation to the varieties $\mathcal{N}_{n,m}$. Assertion (ii) of the next proposition, which is an analogue in arbitrary characteristic of [5, Cor. 4.3], says in particular that $W_{n,m}$ is the transmuted variety of $\mathcal{N}_{n,m}$.

Proposition 1.

(i) *If $m \geq n-1$, then $\overline{\varphi}_{r,s,n,m} : \mathrm{Mat}_{rn} \times \mathrm{Mat}_{ns} \times \mathrm{Mat}_n \to k^n \times \mathrm{Mat}_{rs} \times \mathrm{Mat}_{rs}^m$ is a GL_n-quotient morphism onto its image.*

(ii) *If $r + s \leq n$ and ν is a partition of n with $\nu_1 \leq m+1$, then $Y_{r,s,n} \times \overline{\mathcal{O}}_\nu$ is a good $\mathrm{GL}_{r,s,n}$-variety and $\varphi_{r,s,n,m} : Y_{r,s,n} \times \overline{\mathcal{O}}_\nu \to \mathrm{Mat}_{rs}^m$ is a GL_n-quotient morphism onto its image.*

Proof. (i). If we apply [10, Prop] to the quiver with two nodes x_1 and x_2 of dimensions 1 and n with s arrows from x_1 to x_2, 1 loop at x_2 and r arrows from x_2 to x_1, then we obtain that the algebra of GL_n-invariants of s vectors, r covectors and 1 matrix is generated by $s_1(X), \ldots, s_n(X)$ and the scalar products $\langle f, X^i v \rangle$, where f is one of the covectors, v is one of the vectors, X is the matrix and i is ≥ 0. Of course we may assume that $i < n$ by the Cayley-Hamilton Theorem. So we obtain the assertion.

(ii). As is well-known Mat_{rn} is a good $\mathrm{GL}_r \times \mathrm{GL}_n$-variety and therefore it is also a good $\mathrm{GL}_{r,s,n}$-variety if we let GL_s act trivially. Similarly, Mat_{ns} is also a good $\mathrm{GL}_{r,s,n}$-variety and Mat_n is a good $\mathrm{GL}_{r,s,n}$-variety if we let $\mathrm{GL}_r \times \mathrm{GL}_s$ act trivially. So, by Mathieu's result on tensor products [3, Cor. 4.2.14], $\mathrm{Mat}_{rn} \times \mathrm{Mat}_{ns} \times \mathrm{Mat}_n$ is a good $\mathrm{GL}_{r,s,n}$-variety. Since $r + s \leq n$, $Y_{r,s,n}$ is a good complete intersection in $\mathrm{Mat}_{rn} \times \mathrm{Mat}_{ns}$. So $(\mathrm{Mat}_{rn} \times \mathrm{Mat}_{ns}, Y_{r,s,n})$ is a good pair of $\mathrm{GL}_{r,s,n}$-varieties by [9, Prop. 1.3b(i)]. Furthermore, $(\mathrm{Mat}_n, \overline{\mathcal{O}}_\nu)$ is a good pair of GL_n-varieties by [9, Thm 2.2a(ii)] and therefore also a good pair of $\mathrm{GL}_{r,s,n}$-varieties if we let $\mathrm{GL}_r \times \mathrm{GL}_s$ act trivially. So $(\mathrm{Mat}_{rn} \times \mathrm{Mat}_{ns} \times \mathrm{Mat}_n, Y_{r,s,n} \times \overline{\mathcal{O}}_\nu)$ is a good pair of $\mathrm{GL}_{r,s,n}$-varieties by [9, Prop 1.3e(i)]. This implies the first assertion and if we combine it with (i) and [9, Prop 1.4a] we obtain the second assertion. $\square$

Proposition 2. *Assume $r + s \leq n$ and let ν be a partition of n with $\nu_1 \leq m+1$. Then $\left(\mathrm{Mat}_{rs}^m, \varphi_{r,s,n,m}(Y_{r,s,n} \times \overline{\mathcal{O}}_\nu) \right)$ is a good pair of $\mathrm{GL}_r \times \mathrm{GL}_s$-varieties.*

Proof. Choose $N \geq (m+1)\max(r, s)$. By the argument in the proof of [5, Thm 5.1] we have $\varphi_{r,s,N,m}(\mathcal{N}_{N,m}) = \mathrm{Mat}_{rs}^m$ and therefore we certainly have $\varphi_{r,s,N,m}(Y_{r,s,N} \times \mathcal{N}_{N,m}) = \mathrm{Mat}_{rs}^m$. In the proof of Proposition 1 we have seen that $(\mathrm{Mat}_{rN} \times \mathrm{Mat}_{Ns} \times \mathrm{Mat}_N, Y_{r,s,N} \times \mathcal{N}_{N,m})$ is a good pair of $\mathrm{GL}_{r,s,N}$-varieties.

So by Proposition 1(i) and [9, Prop. 1.4(a)]

(a). $(\overline{\varphi}_{r,s,N,N-1}(\mathrm{Mat}_{rN} \times \mathrm{Mat}_{Ns} \times \mathrm{Mat}_N), \mathrm{Mat}_{rs}^m)$ is a good pair of $\mathrm{GL}_r \times \mathrm{GL}_s$-varieties.

Put $Z_{N,n} = \{(B, X) \in \mathrm{Mat}_{Ns} \times \mathrm{Mat}_N \mid \mathrm{rk}(B|X) \leq n\}$. If we identify $\mathrm{Mat}_{Ns} \times \mathrm{Mat}_N$ with $\mathrm{Mat}_{N,s+N}$, then $(\mathrm{Mat}_{Ns} \times \mathrm{Mat}_N, Z_{N,n})$ is a good pair of $\mathrm{GL}_N \times \mathrm{GL}_{s+N}$-varieties by [9, Prop. 1.4(c)]. By [3, Cor. 4.2.15] it is then a good pair of $\mathrm{GL}_N \times (\mathrm{GL}_s \times \mathrm{GL}_N)$-varieties and by [3, Cor. 4.2.14] it is then also a good pair of $\mathrm{GL}_s \times \mathrm{GL}_N$-varieties if we let GL_N act diagonally. It will also be a good pair of $\mathrm{GL}_{r,s,N}$-varieties if we let GL_r act trivially. So by [9, Prop 1.3e(i)] $(\mathrm{Mat}_{rN} \times \mathrm{Mat}_{Ns} \times \mathrm{Mat}_N, \mathrm{Mat}_{rN} \times Z_{N,n})$ is a good pair of $\mathrm{GL}_{r,s,N}$-varieties. It now follows from [9, Prop. 1.4a] that

(b). $(\overline{\varphi}_{r,s,N,N-1}(\mathrm{Mat}_{rN} \times \mathrm{Mat}_{Ns} \times \mathrm{Mat}_N), \overline{\varphi}_{r,s,N,N-1}(\mathrm{Mat}_{rN} \times Z_{N,n}))$ is a good pair of $\mathrm{GL}_r \times \mathrm{GL}_s$-varieties.

Let $(e_1, \ldots, e_N)$ be the standard basis of k^N and let $(A, B, X) \in \mathrm{Mat}_{rN} \times Z_{N,n}$. Then $\dim(\mathrm{Im}(B) + \mathrm{Im}(X)) \leq n$, so for some $g \in \mathrm{GL}_N$ we have $\mathrm{Im}(gB) + \mathrm{Im}(gX) = \mathrm{Im}(gB) + \mathrm{Im}(gXg^{-1}) \subseteq \{e_1, \ldots, e_n\}$. Write

$$g \cdot A = [\, A_1 \ \ A_2 \,], \ g \cdot X = \begin{bmatrix} X_1 & X_2 \\ 0 & 0 \end{bmatrix}, \ g \cdot B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

$A_1 \in \mathrm{Mat}_{rn}, X_1 \in \mathrm{Mat}_n, B_1 \in \mathrm{Mat}_{ns}$. Then a simple computation shows that $\overline{\varphi}_{r,s,N,N-1}(A, B, X) = \overline{\varphi}_{r,s,n,N-1}(A_1, B_1, X_1)$, so

(c). $\overline{\varphi}_{r,s,N,N-1}(\mathrm{Mat}_{rN} \times Z_{N,n}) = \overline{\varphi}_{r,s,n,N-1}(\mathrm{Mat}_{rn} \times \mathrm{Mat}_{ns} \times \mathrm{Mat}_n)$,

since the inclusion $\supseteq$ is obvious. In the proof of Proposition 1 we saw that $(\mathrm{Mat}_{rn} \times \mathrm{Mat}_{ns} \times \mathrm{Mat}_n, Y_{r,s,n} \times \overline{\mathcal{O}}_\nu)$ is a good pair of $\mathrm{GL}_{r,s,n}$-varieties. So by [9, Prop. 1.4a] we have

(d). $(\overline{\varphi}_{r,s,n,N-1}(\mathrm{Mat}_{rn} \times \mathrm{Mat}_{ns} \times \mathrm{Mat}_n), \varphi_{r,s,n,m}(Y_{r,s,n} \times \overline{\mathcal{O}}_\nu))$ is a good pair of $\mathrm{GL}_r \times \mathrm{GL}_s$-varieties.

Combining (a)-(d) and [9, Lem. 1.3a(ii)] we obtain the assertion. $\qquad \square$

Remarks 1. 1. Similar as in the proof of Proposition 2, one can show that for r and s arbitrary $(\mathrm{Mat}_{rs} \times \mathrm{Mat}_{rs}^m, \varphi_{r,s,n,m}(\mathrm{Mat}_{rn} \times \mathrm{Mat}_{ns} \times \overline{\mathcal{O}}_\nu))$ is a good pair of $\mathrm{GL}_r \times \mathrm{GL}_s$-varieties.
2. The result [9, Thm 2.2a(ii)] can also be deduced from [20, Thm. 4.3] in combination with [3, Ex. 4.2.E.2]. The point is that the splitting from [20] is easily seen to be B-canonical.

By Proposition 1(ii) we have $W_{n,m} \cong Y_{r,s,n} \times^{\mathrm{GL}_n} \mathcal{N}_{n,m} := (Y_{r,s,n} \times \mathcal{N}_{n,m})//\mathrm{GL}_n$. It is well-known that the formal character of $Y_{r,s,n}$ is independent of the characteristic (this can also be deduced from the formula in [9, Prop1.3b(ii)]). So by [18, Thm 6.3] and [14, Thm 9] (see also [5, Thm 3.3]) the sections in a good $\mathrm{GL}_{r,s,n}$-filtration of $k[Y_{r,s,n}]$ are precisely the induced $\mathrm{GL}_{r,s,n}$-modules $\nabla_{\mathrm{GL}_r}(-\mu^{\mathrm{rev}}) \otimes \nabla_{\mathrm{GL}_s}(\lambda) \otimes \nabla_{\mathrm{GL}_n}([\mu, \lambda])$, each occurring once, where λ and μ are partitions with $l(\mu) \leq r$ and $l(\lambda) \leq s$.

Now if V is a good GL_n-variety, then $Y_{r,s,n} \times^{\mathrm{GL}_n} V$ is a good $\mathrm{GL}_r \times \mathrm{GL}_s$-variety by [9, Prop 1.2e(iii)] and, by the above and a simple character calculation, the good filtration multiplicity of $\nabla_{\mathrm{GL}_r}(-\mu^{\mathrm{rev}}) \otimes \nabla_{\mathrm{GL}_s}(\lambda)$ in $k[Y_{r,s,n} \times^{\mathrm{GL}_n} V]$ is equal to that of $\nabla_{\mathrm{GL}_n}([\lambda, \mu])$ in $k[V]$. Note here that $\nabla_{\mathrm{GL}_n}([\mu, \lambda])^* \cong \Delta_{\mathrm{GL}_n}([\lambda, \mu])$. Loosely spoken, each copy of $\nabla_{\mathrm{GL}_n}([\lambda, \mu])$ in $k[V]$ is replaced by $\nabla_{\mathrm{GL}_r}(-\mu^{\mathrm{rev}}) \otimes$

$\nabla_{\mathrm{GL}_s}(\lambda)$ if $l(\mu) \leq r$ and $l(\lambda) \leq s$ and removed otherwise. We can apply this to $V = \mathcal{N}_{n,m}$.

If we give the piece of $k[\mathrm{Mat}_{rs}^m]$ of multidegree ν total degree $\sum_{i=1}^m \nu_i i$, then the vanishing ideals of the varieties $W_{n,m}$ are graded, so their coordinate rings will inherit the above total grading. The aforementioned equalities of good filtration multiplicities for $k[\mathcal{N}_{n,m}]$ and $k[W_{n,m}]$ are then in fact equalities of graded good filtration multiplicities. Furthermore, the graded dimension of $k[\mathcal{N}_{n,m}]_{[\lambda,\mu]}^{U_n}$ is increasing in m, and by the above it is also increasing in n, since $W_{n,m} \subseteq W_{N,m}$ whenever $N \geq n$. It follows that the graded dimension of $k[\mathcal{N}_n]_{[\lambda,\mu]}^{U_n}$ is increasing in n. This was observed by Brylinsky in [5].

The theorem below says that to find finite spanning sets for the highest weight vectors in the coordinate ring of the GL_n-variety $\mathcal{N}_{n,m}$, it is enough to do this for the $\mathrm{GL}_r \times \mathrm{GL}_s$-variety Mat_{rs}^m. We note that, since $k[\mathrm{Mat}_{rs}^m]$ has a good filtration and its formal character is independent of the characteristic, the good filtration multiplicity $\dim k[\mathrm{Mat}_{rs}^m]_{(-\mu^{\mathrm{rev}},\lambda)}^{U_r \times U_s}$ of $\nabla_{\mathrm{GL}_r}(-\mu^{\mathrm{rev}}) \otimes \nabla_{\mathrm{GL}_s}(\lambda)$ in $k[\mathrm{Mat}_{rs}^m]$ is independent of the characteristic of k. A simple character calculation combined with [19, I.7.10(b)] shows that the multigraded good filtration multiplicity of $\nabla_{\mathrm{GL}_r}(-\mu^{\mathrm{rev}}) \otimes \nabla_{\mathrm{GL}_s}(\lambda)$ in $k[\mathrm{Mat}_{rs}^m]$ is $s_\lambda * s_\mu(z_1, \ldots, z_m)$, where s_λ is the Schur function associated to λ, $*$ denotes the internal product of Schur functions and z_i is a grading variable for the i-th matrix component. So this multiplicity is 0 if $|\lambda| \neq |\mu|$ or if $s_\lambda * s_\mu$ only contains Schur functions associated to partitions of length $> m$.

Theorem 1. *Let $\chi = [\lambda, \mu]$ be a dominant weight in the root lattice, $l(\mu) \leq r$, $l(\lambda) \leq s$, $r + s \leq n$, and let ν be a partition of n with $\nu_1 \leq m + 1$. Then the pull-back*

$$k[\mathrm{Mat}_{rs}^m]_{(-\mu^{\mathrm{rev}},\lambda)}^{U_r \times U_s} \to k[\overline{\mathcal{O}}_\nu]_\chi^{U_n}$$

along $\varphi_{r,s,n,m} : \overline{\mathcal{O}}_\nu \to \mathrm{Mat}_{rs}^m$ is surjective, and in case $\overline{\mathcal{O}}_\nu = \mathcal{N}_{n,m}$ and $n \geq (m+1)\max(r,s)$ it is an isomorphism.

Proof. For a matrix M denote by $M_{r\rfloor,\lfloor s}$ the lower left $r \times s$ corner of M and define $M_{r\rfloor,r\rfloor}$ and $M_{\lfloor s,\lfloor s}$ similarly. Then we have

$$(SXS^{-1})_{r\rfloor,\lfloor s} = S_{r\rfloor,r\rfloor} X_{r\rfloor,\lfloor s} (S_{\lfloor s,\lfloor s})^{-1}$$

and therefore

$$\varphi_{r,s,n,m}(SXS^{-1}) = S_{r\rfloor,r\rfloor} \varphi_{r,s,n,m}(X)(S_{\lfloor s,\lfloor s})^{-1}$$

for any $X \in \mathrm{Mat}_n$ and any upper triangular $S \in \mathrm{GL}_n$. So indeed the pull-back along $\varphi_{r,s,n,m}$ maps highest weighty vectors to highest weight vectors and it is an easy exercise to see that the weights correspond as stated in the theorem.

Since $(\mathcal{N}_{n,m}, \overline{\mathcal{O}}_\nu)$ is a good pair of GL_n varieties by [9, Thm. 2.1c, Lem. 1.3a(ii)] we may assume $\overline{\mathcal{O}}_\nu = \mathcal{N}_{n,m}$. By the discussion before the theorem, based on Proposition 1, we know that the good filtration multiplicity of $\nabla_{\mathrm{GL}_r}(-\mu^{\mathrm{rev}}) \otimes \nabla_{\mathrm{GL}_s}(\lambda)$ in $k[W_{nm}]$ is equal to that of $\nabla_{\mathrm{GL}_n}([\lambda,\mu])$ in $k[\mathcal{N}_{n,m}]$. Put differently, we know that $k[W_{nm}]_{(-\mu^{\mathrm{rev}},\lambda)}^{U_r \times U_s}$ and $k[\mathcal{N}_{n,m}]_\chi^{U_n}$ have the same dimension. As we have seen before, $\varphi_{r,s,n,m}(\mathcal{N}_{n,m})$ is dense in W_{nm}, so the pull-back

$k[W_{nm}] \to k[\mathcal{N}_{n,m}]$ along $\varphi_{r,s,n,m}$ is injective and induces an isomorphism between $k[W_{nm}]^{U_r \times U_s}_{(-\mu^{\mathrm{rev}}, \lambda)}$ and $k[\mathcal{N}_{n,m}]^{U_n}_\chi$. By the argument in the proof of [5, Thm 5.1] we have $\varphi_{r,s,n,m}(\mathcal{N}_{n,m}) = \mathrm{Mat}^m_{rs}$ if $n \geq (m+1) \max(r,s)$, which gives us the final assertion. So it suffices to show that the restriction $k[\mathrm{Mat}^m_{rs}]^{U_r \times U_s}_{(-\mu^{\mathrm{rev}}, \lambda)} \to k[W_{nm}]^{U_r \times U_s}_{(-\mu^{\mathrm{rev}}, \lambda)}$ is surjective and this follows from Proposition 2. $\qquad\square$

Remarks 2. 1. Assume $m = n - 1$. If $f \in k[\mathrm{Mat}^m_{rs}]^{U_r \times U_s}_{(-\mu^{\mathrm{rev}}, \lambda)}$ is homogeneous for the total grading defined above, then the pull-back of f along $\varphi_{r,s,n,m}$: $\mathcal{N}_n \to \mathrm{Mat}^m_{rs}$ has an obvious lift to $k[\mathrm{Mat}_n]^{U_n}_{[\lambda, \mu]}$, namely the pull-back along $\varphi_{r,s,n,m}$: $\mathrm{Mat}_n \to \mathrm{Mat}^m_{rs}$. This follows from the fact that the displayed formulas at the beginning of the proof of Theorem 1 hold for any $X \in \mathrm{Mat}_n$. So if we have a spanning set of $k[\mathcal{N}_n]^{U_n}_{[\lambda, \mu]}$ which is pulled back from $k[\mathrm{Mat}^m_{rs}]^{U_r \times U_s}_{(-\mu^{\mathrm{rev}}, \lambda)}$ along $\varphi_{r,s,n,m}$, then we are always in the situation to apply Lemma 1.
2. With the total grading of $k[\mathrm{Mat}^m_{rs}]$ defined above the pull-back along $\varphi_{r,s,n,m}$: $\mathcal{N}_n \to \mathrm{Mat}^m_{rs}$ is a homomorphism of graded vector spaces. By [2, Thm. 2.14] and the independence of the characteristic of the graded formal character of $k[\mathcal{N}_n]$, the good filtration multiplicity of $\nabla_{\mathrm{GL}_n}([\lambda, \mu])$ in the degree d piece of $k[\mathcal{N}_n]$ is the same for all $n \geq l(\lambda) + l(\mu) + d - t$, where $t = |\lambda| = |\mu|$. From this, Theorem 1 and the fact that the graded dimension of $k[\mathcal{N}_{n,m}]^{U_n}_{[\lambda, \mu]}$ is increasing in m and n it follows that the pull-back $k[\mathrm{Mat}^{n-1}_{rs}]^{U_r \times U_s}_{(-\mu^{\mathrm{rev}}, \lambda)} \to k[\mathcal{N}_n]^{U_n}_{[\lambda, \mu]}$ will be an isomorphism in degree d if $n \geq l(\lambda) + l(\mu) + d - t$.

The space $\mathrm{Mat}^m_{rs} = \mathrm{Mat}_{rs} \otimes k^m$ has an extra action of the group GL_m which commutes with the action of $\mathrm{GL}_r \times \mathrm{GL}_s$. For convenience we choose the action induced by the action $g \cdot v = vg^{-1}$ on k^m, where v is considered as a row vector. If we would have used the more obvious action $g \cdot v = gv$ on k^m, then this would amount to twisting the above action with the inverse transpose.

Let λ be a partition of $t \leq r$ with $l(\lambda) \leq s$ and let T_λ be the tableau of shape λ defined in Section 1. For T a tableau of shape λ with entries $\leq m$ we define the semi-invariant $u_T \in k[\mathrm{Mat}^m_{rs}]$ by

$$(A_1, \ldots, A_m) \mapsto$$

$$\sum_S \det \left(A_{S_{11}} e_1 | \cdots | A_{S_{1\lambda_1}} e_1 | \cdots | A_{S_{l(\lambda)1}} e_{l(\lambda)} | \cdots | A_{S_{l(\lambda)\lambda_{l(\lambda)}}} e_{l(\lambda)} \right)_{t]}$$

and the semi-invariant $v_T \in k[\mathrm{Mat}^m_{sr}]$ by

$$(A_1, \ldots, A_m) \mapsto$$

$$\sum_S \det \left(A'_{S_{11}} e_s | \cdots | A'_{S_{1\lambda_1}} e_s | \cdots | A'_{S_{l(\lambda)1}} e_{s-l(\lambda)+1} | \cdots | A'_{S_{l(\lambda)\lambda_{l(\lambda)}}} e_{s-l(\lambda)+1} \right)_{\lfloor t},$$

where the sums are over all tableaux S in the orbit of T under the column stabiliser of T_λ, the subscripts "$t]$" and "$\lfloor t$" mean that we take the last resp. first t rows, the S_{ij} denote the entries of S, the e_i are the standard basis vectors of k^s, and A'_i denotes the transpose of A_i.

Theorem 2. *Let λ be a partition of $t \leq r$ with $l(\lambda) \leq s$ and $\lambda_1 \leq m$. Then*

(i) *the u_T with T row semi-standard form a basis of $k[\mathrm{Mat}_{rs}^m]_{(-(1^t)^{\mathrm{rev}}, \lambda)}^{U_r \times U_s}$,*

(ii) *the v_T with T row semi-standard form a basis of $k[\mathrm{Mat}_{sr}^m]_{(-\lambda^{\mathrm{rev}}, 1^t)}^{U_s \times U_r}$*

and both vector spaces are, with the GL_m-action defined above, isomorphic to the Weyl module of highest weight λ'.

Proof. (i). Put $F = k^m$, let $(f_1, \ldots, f_m)$ be the standard basis of F and put $\bigwedge^\lambda F = \bigwedge^{\lambda_1} F \otimes \cdots \otimes \bigwedge^{\lambda_{l(\lambda)}} F$. For S a tableau of shape λ with entries $\leq m$ we put

$$f_S = f_{S_{11}} \wedge \cdots \wedge f_{S_{1\lambda_1}} \otimes \cdots \otimes f_{S_{l(\lambda)1}} \wedge \cdots \wedge f_{S_{l(\lambda)\lambda_{l(\lambda)}}}.$$

Then the f_S with the rows of S strictly increasing form a basis of $\bigwedge^\lambda F$. From the anti-symmetry properties of the f_S it is clear that there exists a unique linear mapping $\psi : \bigwedge^\lambda F \to k[\mathrm{Mat}_{rs}^m]$ with $\quad \psi(f_S) =$

$$(A_1, \ldots, A_m) \mapsto \det \left(A_{S_{11}} e_1 | \cdots | A_{S_{1\lambda_1}} e_1 | \cdots | A_{S_{l(\lambda)1}} e_{l(\lambda)} | \cdots | A_{S_{l(\lambda)\lambda_{l(\lambda)}}} e_{l(\lambda)} \right)_{t\rfloor}$$

for all tableaux S of shape λ with entries $\leq m$. Furthermore, it is easy to check that ψ is GL_m-equivariant and that the u_T, T row semi-standard are the images of the Carter-Lustig basis elements of the Weyl module of highest weight λ' inside $\bigwedge^\lambda F$, see [13, 5.3b] and [6, Thm 3.5]. So to prove (i) and the final assertion in case (i) it suffices to show that ψ is injective and $k[\mathrm{Mat}_{rs}^m]_{(-(1^t)^{\mathrm{rev}}, \lambda)}^{U_r \times U_s}$ has dimension equal to that of the Weyl module of highest weight λ'. Since the space of highest weight vectors has dimension $s_{1^t} * s_\lambda(1, \ldots, 1) = s_{\lambda'}(1, \ldots, 1)$ (m ones) the latter is indeed true, so it remains to prove the injectivity of ψ.

To prove this will associate to each tableau T of shape λ with entries $\leq m$ and strictly increasing rows an m-tuple of $r \times s$-matrices $A(T)$ such that $\psi(f_S)(A(T))_{S,T}$ is the identity matrix. We define $A(T)$ as follows

$$A(T)_{T_{ij}}(e_i) = e_{(T_\lambda)_{ij}} \quad \text{and} \quad A(T)_h(e_i) = 0 \text{ if } h \notin i\text{-th row of } T \text{ or } l(\lambda) < i \leq s,$$

where we denote the standard basis vectors of $k^{\max(r,s)}$ by $e_1, \ldots, e_{\max(r,s)}$.[2] Then clearly $\psi(f_T)(A(T)) = 1$. Now assume $S \neq T$. Then $S_{ij} \neq T_{ij}$ for certain i, j, so S_{ij} does not occur in the i-th row of T. So $A(T)_{S_{ij}}(e_i) = 0$ and therefore $\psi(f_S)(A(T)) = 0$.

(ii). Let $\Phi : k[\mathrm{Mat}_{rs}^m] \to k[\mathrm{Mat}_{sr}^m]$ be the algebra isomorphism corresponding to vector space isomorphism $\mathrm{Mat}_{sr}^m \to \mathrm{Mat}_{rs}^m$ induced by the vector space isomorphism $A \mapsto P_1 A' P_2^{-1} : \mathrm{Mat}_{sr} \to \mathrm{Mat}_{rs}$, where $P_1 \in \mathrm{GL}_r$ and $P_2 \in \mathrm{GL}_s$ are the permutation matrices which are 1 on the anti-diagonal and 0 elsewhere. Then $\Phi(k[\mathrm{Mat}_{rs}^m]_{(-(1^t)^{\mathrm{rev}}, \lambda)}^{U_r \times U_s}) = k[\mathrm{Mat}_{sr}^m]_{(-\lambda^{\mathrm{rev}}, 1^t)}^{U_s \times U_r}$ and $\Phi(u_T) = \pm v_T$. So (ii) follows from (i). Furthermore, Φ is GL_m-equivariant, so the final assertion also applies to (ii). $\qquad\square$

Remarks 3. 1. If λ or μ is a row one can easily find bases of $k[\mathrm{Mat}_{sr}^m]_{(-\mu^{\mathrm{rev}}, \lambda)}^{U_s \times U_r}$. In this case the GL_m-module structure is that of the induced module of highest weight λ. Unlike the case that λ or μ is a column, the pull-backs of these bases to the nilpotent cone are always bases of $k[\mathcal{N}_n]_{[\lambda, \mu]}^{U_n}$. This can be deduced from

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[2]The reader may consider k^r as a subspace of k^s if $r \leq s$ and conversely otherwise.

the proof of [24, Thm. 2]. For example, for the weight $(-\lambda^{\mathrm{rev}}, 1^t)$, $l(\lambda) \leq m$, one obtains a basis by taking the "left anti-canonical bideterminants" $(\tilde{T}_\lambda \,|\, T)$, T semi-standard of shape λ with entries $\leq m$, on the $r \times m$ matrix obtained by taking the first column of each matrix component of $\underline{A} \in \mathrm{Mat}_{rs}^m$. Here $\tilde{T}_\lambda$ is the anti-canonical tableau denoted by T_λ in [24]. Our results on the GL_m-module structure when λ or μ is a row or a column are in accordance with [1, Sect. III].

2. Combining Theorem 2 and Theorem 1 we obtain spanning sets for the spaces $k[\mathcal{N}_n]_\chi^{U_n}$, where χ is of the form $[\lambda, 1^t]$ or $[1^t, \lambda]$, i.e. for weights χ with $\chi_n \geq -1$ or with $\chi_1 \leq 1$. Assume $\mathrm{char}(k) = 0$. Then the weights χ with $\chi_n \geq -1$ are related to the coinvariant ring C_W of $W = \mathrm{Sym}_n$ via the generalised Chevalley Restriction Theorem as follows:

$$k[\mathcal{N}_n]_\chi^{U_n} \cong \mathrm{Mor}_{\mathrm{GL}_n}(\mathcal{N}_n, L(\chi)^*) \cong \mathrm{Mor}_W(\mathcal{N}_n \cap \mathfrak{t}, L(\chi)_0^*) \cong \mathrm{Hom}_W(L(\chi)_0, C_W) \,.$$

Here $\mathcal{N}_n \cap \mathfrak{t}$ is the scheme-theoretic intersection of $\mathcal{N}_n$ and the vector space of diagonal $n \times n$-matrices $\mathfrak{t}$. In fact one can replace $\mathcal{N}_n$ by an arbitrary nilpotent orbit closure $\overline{\mathcal{O}}_\nu$ and C_W by the corresponding coinvariant ring, see [4]. This means in particular that the graded dimension of $k[\overline{\mathcal{O}}_\nu]_\chi^{U_n}$ is given by $\tilde{K}_{\overline{\lambda}',\nu'}(t)$, where $\overline{\lambda} = \chi + \mathbf{1}_n$, $\mathbf{1}_n$ the all-one vector of length n and $\tilde{K}_{\overline{\lambda}',\nu'}(t) = t^{n(\nu')} K_{\overline{\lambda}',\nu'}(t^{-1})$, $K_{\overline{\lambda}',\nu'}(t)$ the Kostka polynomial, as in [19, p. 248], see e.g. [11].

3. The dimension of the lowest degree piece of $k[\mathcal{N}_n]_\chi^{U_n}$ need not be one: for $\chi = (3, 3, 0, -2, -2, -2)$, the lowest degree is 9 and the piece of degree 9 has dimension 2. For weights of the form $[\lambda, 1^t]$, $[1^t, \lambda]$, $[t, \lambda]$ and $[\lambda, t]$ the dimension of the lowest degree piece is always one. In the first case this follows from the link with the coinvariant algebra mentioned above. By going to bigger n the lowest degree of $k[\mathcal{N}_n]_{[\lambda,\mu]}^{U_n}$ may drop: for $\lambda = (4, 4, 4)$ and $\mu = (3, 3, 3, 3)$ the lowest degree is 18 for $n = 7$ and 17 for $n = 8$. All this can be calculated with the computer using the Lascoux-Schutzenberger charge on tableaux.

3. Coinvariants for Young subgroups and highest weight vectors in characteristic 0

In this section we want to give bases for all the spaces highest weight vectors in $k[\mathrm{Mat}_{rs}^m]$. We will always assume that k has characteristic 0.

3.1. **Representations of the symmetric group.** We give a short account of Donin's results [7] on the representations of the symmetric group. He gave certain explicit bases for Hom spaces between skew Specht modules which are useful for the purpose of finding natural spanning sets for the highest weight vectors in $k[\mathcal{N}_n]$. We drop the assumption that k is algebraically closed. Let G be a finite group and let $A = kG$ be its group algebra. It has the obvious $\mathbb{Q}$-form $A_\mathbb{Q} = \mathbb{Q}G$. Denote the symmetric bilinear form on A for which the group elements form an orthonormal basis by $(-, -)$. Since its restriction to $A_\mathbb{Q}$ is positive definite, its restriction to any $\mathbb{Q}$-defined subspace of A will be nondegenerate. Let $a \mapsto a^*$ be the anti-involution of A which extends the inversion of G. Then we have

$$(ab, c) = (a, cb^*) \text{ and } (ab, c) = (b, a^*c)$$

for all $a, b, c \in A$. To deal with Hom spaces between ideals of A generated by elements that need not be idempotents we need the following lemma.

Lemma 2. *Let $a \in A$ and let M be an A-module.*

(i) *The map $\varphi : x \otimes y \mapsto x^* y : Aa \otimes M \to a^* M$ restricts to an isomorphism $(Aa \otimes M)^G \xrightarrow{\sim} a^* M$. The inverse is given by $\psi : c \mapsto \frac{1}{|G|} \sum_{g \in G} g \otimes gc$.*

(ii) *If $a \in A_{\mathbb{Q}}$, then the composite of ψ with the G-module isomorphism $x \otimes y \mapsto (z \mapsto (x, z)y) : Aa \otimes M \to \mathrm{Hom}(Aa, M)$ maps $c \in a^* M$ to the "right multiplication" by $\frac{1}{|G|}c$.*

(iii) *If $a \in A_{\mathbb{Q}}$, then $Aa = Aa^* a$.*

Proof. (i). Clearly, $\varphi \circ \psi = \mathrm{id}$. Furthermore, we have for all $x, y \in A$ and $z \in M$

$$\sum_{g \in G} gxy \otimes gz = \sum_{g \in G} gy \otimes gx^* z \,.$$

So if $x \in a^* M$, then $\psi(x) \in (Aa \otimes M)^G$. Now $(Aa \otimes M)^G$ is spanned by elements of the form $c = \sum_{g \in G} gxa \otimes gy$, $x \in A$, $y \in M$, and for such a c we have $\psi(\varphi(c)) = \psi(|G|(xa)^* y) = \sum_{g \in G} g \otimes g(xa)^* y = \sum_{g \in G} gxa \otimes gy = c$.

(ii). First note that the given map from $Aa \otimes M$ to $\mathrm{Hom}(Aa, M)$ is obtained by combining the standard isomorphism $(Aa)^* \otimes M \xrightarrow{\sim} \mathrm{Hom}(Aa, M)$ with the isomorphism $x \mapsto (x, -) : Aa \xrightarrow{\sim} (Aa)^*$, so it is indeed an isomorphism. Now we compose ψ with this isomorphism. Then $c \in a^* M$ goes to the map $z \mapsto \frac{1}{|G|} \sum_{g \in G} (g, z)gc = z \frac{1}{|G|}c$.

(iii). Let ρ_a denote the right multiplication by a. Then $\rho_{a^*} = \rho_a'$, the transpose of ρ_a with respect to the form $(-, -)$. So $Aa^* a = \mathrm{Im}(\rho_a \rho_a') = \mathrm{Im}(\rho_a) = Aa$. Here the second equality follows from the corresponding equality on $A_{\mathbb{Q}}$ on which our form is positive definite. $\square$

From now on G will be the symmetric group Sym_t of rank t. To describe certain Hom spaces and certain subspaces of A it will turn out to be useful to use bijections between skew diagrams. We call such bijections *diagram mappings*. If we fix skew diagrams E and F, then the elements of G are in one-one correspondence with diagram mappings $F \to E$ as follows. If $\alpha : F \to E$ is a diagram mapping, then the corresponding element of G sends for any box x of F the number of T_F in x to the number of T_E in $\alpha(x)$. If we fix only one skew diagram E, then we can identify the elements of G with t-tableaux of shape E by replacing (E, F) above by (Δ_t, E) and use the fact that t-tableaux can be identified with diagram mappings $E \to \Delta_t$. So the first correspondence is $g \mapsto \alpha_g = T_E^{-1} \circ g \circ T_F$ and the second one is $g \mapsto g \circ T_E$. For T a t-tableau of shape F we will also denote $T_E^{-1} \circ T$ by α_T.

As is well known one can associate the so-called skew Specht modules to skew diagrams, just like one can associate Specht modules to ordinary Young diagrams. These skew Specht modules are in general not irreducible, in fact they include the Young permutation modules. We briefly recall the construction. If E is a skew Young diagram with t boxes, then we can form the row symmetriser $e_2 = \sum_g g \in A_{\mathbb{Q}}$ where the sum is over the row stabiliser of T_E in G, and the column anti-symmetriser $e_1 = \sum_g \mathrm{sgn}(g)g \in A_{\mathbb{Q}}$ where the sum is over

the column stabiliser of T_E in G. The product $e = e_1 e_2$ is then called the *Young symmetriser* associated to the skew diagram E. Unlike in the case of ordinary Young diagrams, the symmetrisers associated to skew diagrams are no longer idempotent up to a scalar multiple, although e_1 and e_2 of course are.

For example, if $E = \begin{array}{cc} \boxed{1} & \boxed{2} \\ \end{array}$, then $\dim \operatorname{span}(e, e^2) = 2$.

The *skew Specht module* associated to E is the module Ae. We have $Ae = Ae_1 e_2 \subseteq Ae_2$ and Ae_2 is the well-known permutation module associated to E. If λ is the partition which contains the row lengths of E in weakly descending order, then Ae_2 is isomorphic to the usual Young permutation module M^λ. For example, if λ is a partition of length l and

$$E = \overbrace{\boxed{}\cdots\boxed{}}^{\lambda_1 \text{ boxes}} \cdots \underbrace{\boxed{}\cdots\boxed{}}_{\lambda_l \text{ boxes}} ,$$

then $e = e_2$ and $Ae = Ae_2 = M^\lambda$. If $g, h \in G$, then $ge_2 = he_2$ if and only if the tableaux of shape E corresponding to g and h are row equivalent. For $g \in G$ and $T = g \circ T_E$ we denote ge_2 by $\{T\}$ and call it a *tabloid* in accordance with [15]. Furthermore, $ge = ge_1 g^{-1} ge_2$ and $\kappa_T = ge_1 g^{-1}$ is the column anti-symmetriser associated to the skew tableau T. So the element $ge = \kappa_T \{T\}$ is the *polytabloid* e_T from [15]. We will denote it by $[T]$. For a t-tableau T of shape E we have $[T] = \sum_{\pi \in C_E} \operatorname{sgn}(\pi)\{T\pi\}$, where $C_E \leq \operatorname{Sym}(E)$ is the column stabiliser of E.

For the remainder of this section E and F are two skew diagrams and $e = e_1 e_2$ and $f = f_1 f_2$ are the corresponding Young symmetrisers. The next lemma says that, just like Specht modules, skew Specht modules could also have been defined by multiplying row symmetrisers and column anti-symmetrisers the other way round.

Lemma 3.
  (i) *We have $Ae_1 e_2 = Ae_2 e_1 e_2$ and $Ae_2 e_1 = Ae_1 e_2 e_1$.*
  (ii) *The maps $x \mapsto xe_1 : Ae_1 e_2 \to Ae_2 e_1$ and $x \mapsto xe_2 : Ae_2 e_1 \to Ae_1 e_2$ are isomorphisms.*

Proof. (i). Since $e_1^* = e_1$ and $e_2^* = e_2$, we have $e^* = e_2 e_1$ and $e^* e$ is a nonzero scalar multiple of $e_2 e_1 e_2$. Similarly for $\tilde{e} = e_2 e_1$ we have that $\tilde{e}^* \tilde{e}$ is a nonzero scalar multiple of $e_1 e_2 e_1$. The assertion now follows from Lemma 2(iii).
(ii). By (i) these maps are surjective, so, for dimension reasons, they must be isomorphisms. $\qquad\square$

Since the elements of G can be considered as diagram mappings : $F \to E$ we get a spanning set of $\operatorname{Hom}_A(Ae, Af) = e^* Af$ which is labelled by diagram mappings : $F \to E$. In particular we think of Ae as spanned by diagram mappings : $E \to \Delta_t$, i.e. t-tableaux of shape E. It is our goal to find a subset of the above spanning set which is a basis for the space $e^* Af$. First we point out

some special cases, then we state it in general in Theorem 3. Let μ be the tuple of row lengths of E, i.e. the weight of S_E. We have for $g, h \in G$ that $e_2 g = e_2 h$ if and only if $S_E \circ \alpha_g = S_E \circ \alpha_h$. We will say that g or $T = g \circ T_F$ *represents* $S_E \circ \alpha_g = S_E \circ \alpha_T$. So the elements $e_2 g$ with g in a set of representatives for the tableaux of shape F and weight μ form a basis of $e_2 A$. Of course we could change the shape F to any other shape with the same number of boxes. More generally, we have for T_1, T_2 t-tableaux of shape F that $e_2\{T_1\} = e_2\{T_2\}$ if and only if $S_E \circ \alpha_{T_1}$ and $S_E \circ \alpha_{T_2}$ are row equivalent. So the elements $e_2\{T\}$ with T in a set of representatives for the row-ordered tableaux of shape F and weight μ form a basis of $e_2 A f_2$. For a tableau T we define the *standard scan* of T to be the sequence of entries of T, read row by row from left to right and top to bottom. We order the row ordered tableaux of shape F as follows. If $S \neq T$ are two such tableaux, then $S < T$ if and only if $\alpha_i < \beta_i$, where i is the first position where the standard scans α and β of S and T differ. The above basis of $e_2 A f_2$ is now also linearly ordered, since we linearly ordered its index set. We extend the above order to a preorder on all tableaux of shape F by defining $S \leq T$ if and only if $\tilde{S} \leq \tilde{T}$, where $\tilde{S}$ and $\tilde{T}$ are the unique row ordered tableaux that are row-equivalent to S resp. T. The proof of the next trivial lemma is left to the reader.

Lemma 4 (cf. [7, Lem 1.2], [15, Lem. 8.2]). *Let $(x_i)_{i \in I}$ be a family of elements of $e_2 A f_2$ and for each i let y_i be the least element from the above basis of $e_2 A f_2$ involved in x_i. If the y_i are distinct, then $(x_i)_{i \in I}$ is linearly independent.*

Lemma 5 (cf. [7], [15, Lem. 8.3]). *Let F be a skew diagram. If S, T are distinct column equivalent tableaux of shape F with S column ordered, then $S < T$.*

Proof. Denote the i-th rows of S and T by S_i and T_i. Choose i minimal with $S_i \neq T_i$. Then we have $S_{ij} \leq T_{ij}$ for all j with at least one inequality strict. So for each r the number of occurrences of integers $\leq r$ in S_i is $\geq$ to that of T_i with at least one inequality strict. So $S < T$. $\square$

As in [15, Thm. 8.4] one can use the previous two lemma's (replace (E, F) by (Δ_t, E)) and an obvious generalisation of the Garnir relations [15, Sect. 7] to prove the well-known result that the polytabloids $[T]$, T a standard tableau of shape E, form a basis of Ae.

Lemma 6 ([7, Lem 2.2] and [8, Prop.]). *Let $\alpha : F \to E$ be a diagram mapping which satisfies*

(a) *The tableau $S_E \circ \alpha$ of shape F is semi-standard.*
(b) *If for $a, b \in F$, $\alpha(b)$ occurs strictly below $\alpha(a)$ in the same column, then b occurs in a strictly lower row than a.*

Then there exists a diagram mapping $\tilde{\alpha} : F \to E$ with $S_E \circ \tilde{\alpha} = S_E \circ \alpha$ satisfying

(b') *If for $a, b \in F$, $\tilde{\alpha}(b)$ occurs strictly below $\tilde{\alpha}(a)$ in the same column, then b occurs in a strictly lower row than a and in a column to the left of a or in the same column.*

Proof. Let $a = (i, j) \in F$ be the first cell in the order of the standard scan such that with $\alpha(a) = (r, s)$ we have $(r + 1, s) \in E$ and $b = \alpha^{-1}(r + 1, s)$ occurs in

a column strictly to the right of a (*). Since $S_E(\alpha(a)) = r$, $S_E(\alpha(b)) = r + 1$, $S_E \circ \alpha$ is semi-standard and α has property (b) we have $b = (i + 1, j_1)$ for some $j_1 > j$, $S_E(\alpha(i, j_2)) = r$ and $S_E(\alpha(i + 1, j_2)) = r + 1$ for all j_2 with $j \leq j_2 \leq j_1$. Now put $b_1 = (i + 1, j)$ and $\beta = \alpha \circ (b, b_1)$, where (b, b_1) is the transposition which swaps b and b_1. Then $S_E \circ \beta = S_E \circ \alpha$. If β does not have property (b'), then the first cell of F in the order of the standard scan that has property (*) for β will be after a. This is clear if with $\alpha(b_1) = (r + 1, s_1)$ we have $(r, s_1) \notin E$. So assume this is not the case and assume $a_1 = \alpha^{-1}(r, s_1)$ occurs before a in the standard scan. Then, by the choice of a, its column index is $> j$. So its row index is $< i$. But then, by the semi-standardness of $S_E \circ \alpha$, its column index is $> j_1$. So a_1 doesn't have the above property for β and this was the only possibility before a. So we can finish by induction. $\qquad\square$

Recall that μ is the tuple of row lengths of E. We will call a semi-standard tableau S of shape F and weight μ *admissible* if $S = S_E \circ \alpha$ for some diagram mapping $\alpha : F \to E$ satisfying the conditions (a) and (b) from Lemma 6. Next we need the notion of a "picture" (we will call it admissible) from [25] which is a generalisation of that of [16]. For this we need two orderings $\leq$ and $\preceq$ on $\mathbb{N} \times \mathbb{N}$ defined by $(p, q) \leq (r, s)$ if and only if $p \leq r$ and $q \leq s$, and $(p, q) \preceq (r, s)$ if and only if $p < r$ or ($p = r$ and $q \geq s$). Note that $\preceq$ is a linear ordering. Recall that skew Young diagrams are by definition subsets of $\mathbb{N} \times \mathbb{N}$. A diagram mapping $\alpha : F \to E$ is called *admissible* if $\alpha : (F, \leq) \to (E, \preceq)$ and $\alpha^{-1} : (E, \leq) \to (F, \preceq)$ are order preserving. So α is admissible if and only if α^{-1} is admissible. In [26, App. 2] it is shown that $\alpha : F \to E$ is admissible if and only if for all $a, b \in F$

(1) $a(E)b \implies \alpha(a)(W, SW)\alpha(b)$,
(2) $a(S)b \implies \alpha(a)(SW, S)\alpha(b)$,
(3) $a(NE)b \implies \alpha(a)(NE, N, NW, W, SW)\alpha(b)$,
(4) $a(SE)b \implies \alpha(a)(SW)\alpha(b)$.

Here the letter combinations E, S, SW etc. in the brackets refer to the usual wind directions and they are mutually exclusive. For example, $a(W)b$ means that a occurs strictly before b in the same row and $a(SW)b$ means that a occurs in a row strictly below b and in a column strictly to the left of b. Furthermore, "$a(A, B)b$" means "$a(A)b$ or $a(B)b$" and similar for more than two wind directions. In [26] it is also pointed out that property (4) actually follows from (1) and (2). Although we will not use this equivalent characterisation, it can be useful to get an idea of what it means for a diagram mapping to be admissible.

Theorem 3 ([7, Thm 2.4], [8, Thm 1])**.**

(i) *The elements $e^*[T]$ with T in a set of representatives of the admissible semi-standard tableaux of shape F and weight μ form a basis of e^*Af.*

(ii) *For every admissible semi-standard tableau S of shape F and weight μ, there is precisely one admissible diagram mapping $\alpha : F \to E$ such that $S = S_E \circ \alpha$ and all admissible diagram mappings occur in this way.*

Proof. Assume $\alpha : F \to E$ is admissible. Then it follows that $S = S_E \circ \alpha$ is ordered, since the ordering $\leq$ is linear on the rows and columns of F. Furthermore, $\alpha^{-1} : E \to F$ is also admissible. From this it follows that if b is strictly below a in the same column of F, then $\alpha(b)$ occurs in a row strictly below

$\alpha(a)$, i.e. S is semi-standard. Since α^{-1} has the analogous property, α has property (b), i.e. S is admissible. The image of the i-th row of E under α^{-1} is $S^{-1}(i)$, and, since the ordering $\le$ is linear on the rows of E, α^{-1} is completely determined by the images of the rows of E under α^{-1}. So for every admissible semi-standard tableau S of shape F and weight μ, there is at most one admissible diagram mapping $\alpha : F \to E$ such that $S = S_E \circ \alpha$. By [25, Thm 1] the number of admissible diagram mappings is equal to $\dim \mathrm{Hom}_A(Ae, Af)$ which is equal to $\dim e^* Af$ by Lemma 2. So to prove (i) and (ii) it suffices to show that the elements given in (i) are linearly independent.

By Lemma 6 we may assume that for all T in the given set of representatives α_T satisfies property (b') from Lemma 6. Let $C_{T_E} \le G$ and $C_F \le \mathrm{Sym}(F)$ be the column stabilisers of T_E and F and let T be as above. Then we have

$$e^*[T] = \sum_{g \in C_{T_E},\, \sigma \in C_F} \mathrm{sgn}(g)\mathrm{sgn}(\sigma)e_2 g\{T\sigma\} = \sum_{\pi \in \tilde{C}_F,\, \sigma \in C_F} \mathrm{sgn}(\pi)\mathrm{sgn}(\sigma)e_2\{T\pi\sigma\}\,,$$

where $\tilde{C}_F = T^{-1}C_{T_E}T = \alpha_T^{-1}C_E\alpha_T \le \mathrm{Sym}(F)$ is the stabiliser of the sets $\alpha_T^{-1}(E^i)$, E^i the i-th column of E. If, for $\pi \in \tilde{C}_F$, $S_E \circ \alpha_{T\pi}$ has a repeated entry in some column, then $\sum_{\sigma \in C_F} \mathrm{sgn}(\sigma)e_2\{T\pi\sigma\} = 0$. By Lemma 4 it suffices to show that $e_2\{T\}$ occurs with strictly positive coefficient in $e^*[T]$ and $e_2\{T\} \le e_2\{T\pi\sigma\}$ for all $\pi \in \tilde{C}_F$ such that $S_E \circ \alpha_{T\pi}$ has no repeated entry in any column, and all $\sigma \in C_F$.

For $\pi \in \tilde{C}_F$ with this property let $\sigma_\pi \in C_F$ be the element such that $S_E \circ \alpha_{T\pi\sigma_\pi}$ is (strictly) column ordered. Then $S_E \circ \alpha_{T\pi\sigma_\pi} < S_E \circ \alpha_{T\pi\sigma}$ for all $\sigma \in C_F \setminus \{\sigma_\pi\}$ by Lemma 5. So it suffices to show that $e_2\{T\}$ occurs with strictly positive coefficient in $e^*[T]$ and that for π as above $e_2\{T\} \le e_2\{T\pi\sigma_\pi\}$. Let $\pi \in \tilde{C}_F$ such that $S_E \circ \alpha_{T\pi}$ has no repeated entry in any column. If $\pi \in C_F$, then $\sigma_\pi = \pi^{-1}$ and, $\mathrm{sgn}(\pi)\mathrm{sgn}(\sigma_\pi)e_2\{T\pi\sigma_\pi\} = e_2\{T\}$. Now assume $\pi \notin C_F$.

We will finish by showing that $e_2\{T\} < e_2\{T\pi\sigma_\pi\}$. Let $a_1 = (i_1, j_1)$ be the first cell of F in the order of the standard scan which is moved to another column by π^{-1}. So a_1 is the first cell whose value $r = S_E(\alpha_T(a_1))$ has moved to another column in $S_E \circ \alpha_T\pi$. First we prove the following claim.

Claim. *If $a = (i, j)$ and $\pi(a)$ are not in the same column, then we have $S_E(\alpha_T(\pi(a))) \ge S_E(\alpha_T(i_1, j))$.*

Proof. Assume a has the stated property. From the definition of a_1 it follows that $\pi(a)$ has row index $\ge i_1$. If $\pi(a)$ has column index $> j$, then the semi-standardness of $S_E \circ \alpha_T$ gives us the result. So we assume now that $\pi(a)$ has column index $< j$. Put $D = \alpha_T^{-1}(D')$, where D' is the column of E to which $\alpha_T(a)$ belongs. Note that since α_T has properties (a) and (b'), the inverse images of the columns of E under α_T are vertical strips (see [19]). Furthermore, they are stable under π. Note also that $S_E(b)$ is the row index of b in E, so a cell of D in a lower row than another cell of D must contain a strictly bigger number. Since the intersection of D with the j-th column of F is not stable under π, it is also not stable under π^{-1}. So for some $b \in D$ in the j-th column of F, $\pi^{-1}(b)$ is not in the j-th column. By the definition of a_1, b has row index $\ge i_1$. So $S_E(\alpha_T(i_1, j)) \le S_E(\alpha_T(b))$, by the semi-standardness of $S_E \circ \alpha_T$. Now

$\pi(a)$ occurs in a row strictly below b, since its column index is $< j$ and D is a vertical strip. So $S_E(\alpha_T(b)) < S_E(\alpha_T(\pi(a)))$. $\qquad\square$

From the claim and the choice of a_1 it immediately follows that $S_E \circ \alpha_T$ and $S_E \circ \alpha_T \pi \sigma_\pi$ have the same first $i_1 - 1$ rows, and

$$S_E(\alpha_T(\pi\sigma_\pi(i_1, j))) \geq S_E(\alpha_T(i_1, j)) \text{ for all } j, \text{ with equality if } j < j_1. \qquad (*)$$

Now let $j_0, \ldots, j_2$ be the positions in the i_1-th row where $S_E \circ \alpha_T$ has an r. By (*) these are the only positions in the i_1-th row where $S_E \circ \alpha_T \pi \sigma_\pi$ could have an r. Note that $j_0 \leq j_1 \leq j_2$. Now let a be any cell of $S_E \circ \alpha_T$ which contains an r such that $\pi^{-1}(a)$ has column index in $\{j_0, \ldots, j_2\}$. If the column index of a is $> j_2$, then, by the semi-standardness of $S_E \circ \alpha_T$, its row index is $< i_1$. So, by the definition of a_1, $\pi^{-1}(a)$ is in the same column as a which is impossible. Now assume $\pi^{-1}(a)$ occurs in a column strictly to the right of a. Put $D = \alpha_T^{-1}(D')$, where D' is the column of E to which $\alpha_T(a)$ belongs. Since D is a vertical strip $\pi^{-1}(a)$ has row-index strictly less than that of a and must contain an number $< r$. So, by the semi-standardness of $S_E \circ \alpha_T$, its row index is $< i_1$. By the definition of a_1, $\pi^{-1}(\pi^{-1}(a))$ is in the same column as $\pi^{-1}(a)$. If its row index would be $\geq i_1$, then D would have to contain another cell than a with an r, since it is a vertical strip. This is impossible, so $\pi^{-1}(\pi^{-1}(a))$ has row index $< i_1$. But then we could keep applying π^{-1} and stay in the same column. This contradicts the fact that π^{-1} has finite order. So if $\pi^{-1}(a)$ has column index in $\{j_0, \ldots, j_2\}$, then the same is true for a. Furthermore, if this were true for a_1, then $\pi^{-1}(a_1)$ would have to occur in a column strictly to the left of a_1. Then it follows from the definition of a_1 that $S_E \circ \alpha_T \pi$ would have two r's in the column containing $\pi^{-1}(a_1)$, contradicting our assumption on π.

So the number of occurrences of r in the i_1-th row of $S_E \circ \alpha_T \pi \sigma_\pi$ is at least one less than in the i_1-th row of $S_E \circ \alpha_T$ and by (*) the number of occurrences of any $r' < r$ in the i_1-th row is the same. So we may finally conclude that $S_E \circ \alpha_T \pi \sigma_\pi > S_E \circ \alpha_T$. $\qquad\square$

Remarks 4. 1. It is easy to see that for any semi-standard tableau S of shape F and weight μ there is a standard tableau T of shape F with $S = S_E \circ T_E^{-1} \circ T$. So the representatives T from Theorem 3 can be chosen standard.

2. Write $E = \nu/\tilde{\nu}$. Using Lemma 6, it is easy to see that an admissible tableau of shape F and weight μ must satisfy the condition from [22, Cor 2] that $\tilde{\nu} + w(T_{\geq j})$ is dominant for all j. Since both sets count the same dimension, the two conditions are equivalent.

3. Donin considers tableaux of shape E as diagram mappings $T : \Delta_t \to E$, where Sym_t acts via $\pi \cdot T = T \circ \pi^{-1}$ and he works with the modules $e^* A$ considered as left Sym_t modules via the inversion. In his approach one has to use the isomorphism $\mathrm{Hom}_A(e^* A, f^* A) \cong f^* Ae$, and think of this space as having a spanning set labelled by diagram mappings $: E \to F$. Furthermore, one then has to replace $(a, b, \alpha(a), \alpha(b))$ by $(\alpha(a), \alpha(b), a, b)$ in property (b) and (b') in Lemma 6.

Of course the previous results are valid for any symmetric group $\mathrm{Sym}(X)$, X a finite subset of $\mathbb{N}$ with t elements. Just redefine T_F by writing by filling in the elements from X in their natural order row by row from left to right and top

to bottom and replace "t-tableau of shape E" by "X-tableau of shape E": this is a tableau whose entries are the elements of X (so its entries are distinct).

For $X \subseteq \{1, \ldots, t\}$ we consider $\mathrm{Sym}(X)$ as a subgroup of Sym_t by letting the permutations from $\mathrm{Sym}(X)$ fix everything outside X. When we apply our previous results to $\mathrm{Sym}(X)$ we use X as an extra subscript when necessary. The group algebra $A_X = k\mathrm{Sym}(X)$ is a subalgebra of A. If D is a skew tableau with $|X|$ boxes, then we denote the Young symmetriser associated to the standard tableau $T_{D,X}$ by $e_{D,X}$.

Let $\nu = (\nu_1, \ldots, \nu_m)$ be an m-tuple of integers ≥ 0 with sum t. For $i \in \{1, \ldots, m\}$, put $\Lambda_i = \{j + \sum_{h=1}^{i-1} \nu_h \,|\, 1 \leq j \leq \nu_i\}$. Then the *Young subgroup* Sym_ν of Sym_t associated to ν is the simultaneous stabiliser of the sets $\Lambda_1, \ldots, \Lambda_m$. So $\mathrm{Sym}_\nu \cong \prod_{i=1}^m \mathrm{Sym}_{\nu_i}$. Let $\lambda \supseteq \mu$ be partitions with $E = \lambda/\mu$. Then there is a 1-1 correspondence between ordered tableaux of shape E with entries $\leq m$ and sequences of partitions $\lambda^0, \ldots, \lambda^m$ with $\mu = \lambda^0 \subseteq \lambda^1 \subseteq \cdots \subseteq \lambda^m = \lambda$. Indeed if P is such a tableau, then $(\mu \cup P^{-1}(\{1, \ldots, i\}))_{1 \leq i \leq m}$ is such a sequence of partitions. Conversely we can construct P from such a sequence: just fill the boxes of λ^i/λ^{i-1} with i's for all $i \in \{1, \ldots, m\}$. So we can express the well-known rule for restricting skew Specht modules to Young subgroups in terms of tableaux P as above. We say that a t-tableau T of shape E *belongs to* P if $T^{-1}(\Lambda_i) = P^{-1}(i)$ for all $i \in \{1, \ldots, m\}$. Then T will be standard if and only if the $T|_{P^{-1}(i)}$ are standard. Every standard tableau of shape E belongs to some ordered tableau of shape E and weight ν. If P is an ordered tableaux of shape E and weight ν, then we define T_P to be the tableau of shape E with $T_P|_{P^{-1}(i)} = T_{P^{-1}(i),\Lambda_i}$. Note that T_P is a standard tableau which belongs to P.

Let P and Q be ordered tableaux of shapes E and F, both of weight $\nu \in \mathbb{Z}^m$. Then a diagram mapping $\alpha : F \to E$ with $P \circ \alpha = Q$ determines an m-tuple of tableaux $(S_{P^{-1}(1)} \circ \alpha_1, \ldots, S_{P^{-1}(m)} \circ \alpha_m)$ (*), where $\alpha_i : Q^{-1}(i) \to P^{-1}(i)$ is the restriction of α to $Q^{-1}(i)$. We will say that α *represents* (*). Notice that all the m-tuples (*) have the same tuple of shapes and the same tuple of weights. We express this by saying that the tuple of tableaux has *shapes determined by Q and weights determined by P*. Similarly, if T is a tableau of shape F which belongs to Q, then we say that T *represents* (*), where $\alpha_i = T_{P^{-1}(i),\Lambda_i}^{-1} \circ T|_{Q^{-1}(i)}$. So if we cut T to pieces according to Q, then $\alpha_i : Q^{-1}(i) \to P^{-1}(i)$ above is just the diagram mapping corresponding to the i-th piece. Note that the "union" of the above α_i is $T_P^{-1} \circ T$.

Let ν be as above. If H is a group and U an H-module, then U_H, sometimes called the space of "coinvariants", is defined as the largest quotient of U which has trivial H-action, i.e. the quotient of U by the subspace spanned by the elements $gx - x$, $x \in U, g \in H$.

Proposition 3. *Assume that E and F are ordinary Young tableaux. Let ν and Sym_ν be as above. Then the canonical images of the elements $[T_P] \otimes [T]$, where for each pair (P, Q) with P and Q ordered tableaux of shapes E and F, both of weight ν, T goes through a set of representatives for the m-tuples of admissible semi-standard tableaux with shapes determined by Q and weights determined by P, form a basis for $(Ae \otimes Af)_{\mathrm{Sym}_\mu}$.*

Proof. Let Ω_E be the set of ordered tableaux of shape E and weight ν. For $P \in \Omega_E$ put $M_P = \bigoplus_{P \in \Omega} \otimes_{i=1}^m A_{\Lambda_i} e_{P^{-1}(i), \Lambda_i}$ and let $\theta_P : M_P \to Ae$ be the linear map which sends $\otimes_{i=1}^m [T_i]$, T_i standard of shape $P^{-1}(i)$ with entries in Λ_i, to $[T]$ where T is the (standard) tableau obtained by piecing the tableaux T_i together according to P. Then it follows from the basis theorem for Ae that $Ae = \bigoplus_{P \in \Omega_E} \theta_P(M_P)$. By [16, Thm. 3.1] and a straightforward induction argument there is a total ordering $P_1 < P_2 < \cdots < P_p$ of Ω_E such that with $N_j = \bigoplus_{h=1}^j \theta_{P_h}(M_{P_h})$ we have that for all $j \in \{1, \ldots, p\}$ N_j is a Sym_ν-submodule and the natural map $\overline{\theta}_{P_j} : M_{P_j} \to N_j/N_{j-1}$ is an isomorphism of Sym_ν-modules. In particular, if T is a t-tableau which belongs to P_j, then $[T] \in N_j$ and the canonical image of $[T]$ in N_j/N_{j-1} is the image of $\otimes_{i=1}^m [T|_{P^{-1}(i)}]$ under $\overline{\theta}_{P_j}$. Similar remarks apply to analogously defined Ω_F and, for $Q \in \Omega_F$, M_Q and θ_Q. So (redefining the P_j) there is a total ordering $(P_1, Q_1) < (P_2, Q_2) < \cdots < (P_{pq}, Q_{pq})$ of $\Omega_E \times \Omega_F$ such that with (redefining) $N_j = \bigoplus_{h=1}^j \theta_{P_h}(M_{P_h}) \otimes \theta_{Q_h}(M_{Q_h})$ we have that for each $j \in \{1, \ldots, pq\}$ N_j is a Sym_ν-submodule and the natural map $\overline{\theta}_{P_j} \otimes \overline{\theta}_{Q_j} : M_{P_j} \otimes M_{Q_j} \to N_j/N_{j-1}$ is an isomorphism of Sym_ν-modules.

Denote for each $P \in \Omega_E$ and $Q \in \Omega_F$ the given set of representative t-tableaux by Γ_{PQ}. Let $\pi_j : N_j \to N_j/N_{j-1}$ be the natural map. By Theorem 3, Lemma 2(i) and the fact that $\overline{\theta}_{P_j} \otimes \overline{\theta}_{Q_j}$ is a homomorphism of Sym_ν-modules, the canonical images of the elements $\pi_j([T_P] \otimes [T])$, $T \in \Gamma_{P_j Q_j}$, in $(N_j/N_{j-1})_{\mathrm{Sym}_\nu}$ form a basis for $(N_j/N_{j-1})_{\mathrm{Sym}_\nu}$. When applying Lemma 2(i) we omitted the sum over $\mathrm{Sym}(\Lambda_i)$ coming from the definition of ψ after moving $e^*_{Q_j^{-1}(i), \Lambda_i}$ to the left as $e_{Q_j^{-1}(i), \Lambda_i}$, since we work with coinvariants rather than invariants. Now the assertion follows by a straightforward induction. $\square$

Remark 5. The result [7, Thm. 3.1] which deals with restriction to Young subgroups is incorrect since it assumes that the $\theta_P(M_P)$ are Sym_ν-submodules.

3.2. **Bases for the highest weight vectors.** We return to the notation of Section 2. In particular m, r, s are fixed integers ≥ 1. For $l \in \{1, \ldots, m\}$ we denote the matrix entry functions of the l-th matrix component on Mat_{rs}^m by $x(l)_{ij}$. For t an integer ≥ 0 let Σ_t be the set of m-tuples $\nu = (\nu_1, \ldots, \nu_m)$ of integers ≥ 0 with sum t. Furthermore, if λ is a partition, then we define $C_\lambda \leq \mathrm{Sym}(\lambda)$ to be the column stabiliser of λ.

In the papers by Donin referred to below the elements $u_{\nu, P, Q, \alpha}$ were not given, but, in the case $m = n - 1$, their pull-backs to Mat_n under the map $\varphi_{r,s,n,n-1} : \mathrm{Mat}_n \to \mathrm{Mat}_{rs}^{n-1}$ from Section 2. Here one should also note that Donin worked with $S(\mathrm{Mat}_n)$ rather than $k[\mathrm{Mat}_n]$, so, after pulling back, $x_{ij} \in \mathrm{Mat}_n^*$ should be replaced by Donin's $e_{ji} \in \mathrm{Mat}_n$.

Theorem 4 (cf. [8, after Thm 3],[7, Prop. 4.1]). *Let* λ, μ *be a partitions of* t *with* $l(\mu) \leq r$ *and* $l(\lambda) \leq s$. *For* $\nu \in \Sigma_t$, P, Q *ordered tableaux of shapes* λ *and* μ, *both of weight* ν *and* $\alpha : \mu \to \lambda$ *a diagram mapping such that* $P \circ \alpha = Q$ *define*

$$u_{\nu, P, Q, \alpha} = \sum_{\pi \in C_\mu, \sigma \in C_\lambda} \mathrm{sgn}(\pi)\mathrm{sgn}(\sigma) \prod_{a \in \mu} x(Q(a))_{r - \pi(a)_1 + 1, \sigma(\alpha(a)_1)},$$

where for $b \in \mu$, b_1 denotes the row index of b in μ and similar for $b \in \lambda$. Then the elements $u_{\nu,P,Q,\alpha}$, where for each P, Q, ν as above α goes through a set of representatives for the m-tuples of admissible semi-standard tableaux with shapes determined by Q and weights determined by P, form a basis of $k[\mathrm{Mat}^m_{rs}]^{U_r \times U_s}_{(-\mu^{\mathrm{rev}}, \lambda)}$.

Proof. Let $V = k^r$ and $W = k^s$ be the natural modules of GL_r and GL_s. Then $\mathrm{Mat}_{rs} = V \otimes W^*$ and $\mathrm{Mat}^*_{rs} = V^* \otimes W$. So $k[\mathrm{Mat}^m_{rs}] = \bigoplus_{t \geq 0} S^t((V^* \otimes W)^m) = \bigoplus_{t \geq 0, \nu \in \Sigma_t} S^\nu(V^* \otimes W) = \bigoplus_{t \geq 0, \nu \in \Sigma_t} ((V^*)^{\otimes t} \otimes W^{\otimes t})_{\mathrm{Sym}_\nu}$, where, for U any vector space $S^\nu(U) = \otimes^m_{i=1} S^{\nu_i}(U)$. Therefore

$$k[\mathrm{Mat}^m_{rs}]^{U_r \times U_s}_{(-\mu^{\mathrm{rev}}, \lambda)} = \bigoplus_{\nu \in \Sigma_t} \left(((V^*)^{\otimes t})^{U_r}_{-\mu^{\mathrm{rev}}} \otimes (W^{\otimes t})^{U_s}_\lambda \right)_{\mathrm{Sym}_\nu}.$$

As is well-known, $((V^*)^{\otimes t})_{-\mu^{\mathrm{rev}}}$ and $(W^{\otimes t})_\lambda$ are the permutation modules associated to μ and λ, and $((V^*)^{\otimes t})^{U_r}_{-\mu^{\mathrm{rev}}}$ and $(W^{\otimes t})^{U_s}_\lambda$ are the Specht modules Ae_μ and Ae_λ, where $A = k\mathrm{Sym}_t$. To each t-tableau T of shape μ we associate the highest weight vector $e_{1,T} v^*_T \in (V^*)^{\otimes t}$, where v^*_T is the basis tensor which has v^*_{r-i+1}'s in the positions which occur as entries in the i-th row, and $e_{1,T}$ is the column anti-symmetriser associated to T. We also associate to each t-tableau of T shape λ the highest weight vector $e_{1,T} w_T \in W^{\otimes t}$, where w_T is the basis tensor which has w_i's in the positions which occur as entries in the i-th row, and again $e_{1,T}$ is the column anti-symmetriser associated to T. Then $[T] \mapsto e_{1,T} v^*_T : Ae_\mu \to ((V^*)^{\otimes t})^{U_r}_{-\mu^{\mathrm{rev}}}$ and $[T] \mapsto e_{1,T} w_T : Ae_\lambda \to (W^{\otimes t})^{U_s}_\lambda$ are isomorphisms. So by Proposition 3 with $E = \lambda$ and $F = \mu$ the canonical images in $M = \left(((V^*)^{\otimes t})^{U_r}_{-\mu^{\mathrm{rev}}} \otimes (W^{\otimes t})^{U_s}_\lambda \right)_{\mathrm{Sym}_\nu}$ of the elements

$$e_{1,T} v^*_T \otimes e_{1,T_P} w_{T_P} = \sum_{\pi \in C_\mu, \sigma \in C_\lambda} \mathrm{sgn}(\pi) \mathrm{sgn}(\sigma) v^*_{T\pi^{-1}} \otimes w_{T_P \sigma^{-1}},$$

where for each P, Q, ν as above T goes through a set of representatives for the m-tuples of admissible semi-standard tableaux with shapes determined by Q and weights determined by P, form a basis of M. Here we put in the inverses for convenience below. Now we change from representative tableaux T to representative diagram mappings α via $\alpha = T_P^{-1} \circ T$ and take basis elements of V and W which occur in the same tensor position together: $v^*_{T\pi^{-1}}$ has $v^*_{r-\pi(a)_1+1}$ in position $T(a)$ and $w_{T_P \sigma^{-1}}$ has $w_{\sigma(b)_1}$ in position $T_P(b)$, and those positions are the same if and only if $b = \alpha(a)$. Finally, $T(a) \in \Lambda_{Q(a)}$, since T belongs to Q. So $v^*_{r-\pi(a)_1+1} \otimes w_{\sigma(\alpha(a))_1}$ becomes $x(Q(a))_{r-\pi(a)_1+1, \sigma(\alpha(a))_1}$. $\square$

The next corollary gives a much simpler (but bigger) spanning set for the space of highest vectors $k[\mathrm{Mat}^m_{rs}]^{U_r \times U_s}_{(-\mu^{\mathrm{rev}}, \lambda)}$. Of course it can, like the above theorem, be combined with Theorem 1 and Lemma 1 to give spanning sets for the vector space $k[\mathcal{N}_n]^{U_n}_{[\lambda, \mu]}$ and the $k[\mathfrak{gl}_n]^{\mathrm{GL}_n}$-module $k[\mathfrak{gl}_n]^{U_n}_{[\lambda, \mu]}$.

Corollary. *Let $\alpha = 1^{\mu_1} 2^{\mu_2} \cdots$ and $\beta = 1^{\lambda_1} 2^{\lambda_2} \cdots$ be the standard scans of S_μ and S_λ. Then the elements*

$$\sum_{\pi \in C_{T_\mu}, \sigma \in C_{T_\lambda}} \mathrm{sgn}(\pi) \mathrm{sgn}(\sigma) \prod_{i=1}^{t} x(\gamma_i)_{r-\alpha_{\pi(i)}+1, \beta_{\sigma(\tau(i))}},$$

where $\gamma \in \{1, \ldots, m\}^t$ and $\tau \in \mathrm{Sym}_t$ form a spanning set of $k[\mathrm{Mat}_{rs}^m]_{(-\mu^{\mathrm{rev}}, \lambda)}^{U_r \times U_s}$.

Proof. In Theorem 4 we take for each ν the bigger spanning set

$$e_{1,S} v_S^* \otimes e_{1,T} w_T = \sum_{\pi \in C_S, \sigma \in C_T} \mathrm{sgn}(\pi) \mathrm{sgn}(\sigma) v_{\pi^{-1}S}^* \otimes w_{\sigma^{-1}T},$$

where S and T are any t-tableaux of shape μ and λ. Write $T = \rho^{-1} T_\mu$, $S = \tau^{-1} T_\lambda$ for $\rho, \tau \in \mathrm{Sym}_t$. Then we get

$$e_{1,S} v_S^* \otimes e_{1,T} w_T = \sum_{\pi \in C_{T_\mu}, \sigma \in C_{T_\lambda}} \mathrm{sgn}(\pi) \mathrm{sgn}(\sigma) \otimes_{i=1}^t v_{r-\alpha_{\pi(\rho(i))}+1}^* \otimes \otimes_{i=1}^t w_{\beta_{\sigma(\tau(i))}},$$

which corresponds to the element

$$\sum_{\pi \in C_{T_\mu}, \sigma \in C_{T_\lambda}} \mathrm{sgn}(\pi) \mathrm{sgn}(\sigma) \prod_{i=1}^{t} x(\gamma_i)_{r-\alpha_{\pi(\rho(i))}+1, \beta_{\sigma(\tau(i))}} \in k[\mathrm{Mat}_{rs}^m],$$

where $\gamma \in \{1, \ldots, m\}^t$ is the tuple with $i \in \Lambda_{\gamma_i}$ for all i. Recall that the Λ_i depend on ν and note that γ determines ν. Now we observe that if we allow arbitrary tuples $\gamma \in \{1, \ldots, m\}^t$ we can take $\rho = \mathrm{id}$. So we obtain the assertion. $\square$

Remarks 6. 1. It is instructive to consider some special cases. For example, in the case $t = m$ and ν the all-one vector, the highest weight vectors of multidegree ν are labelled by pairs (P, Q) of standard tableaux of shape λ and μ. Another example is the case that λ consists of one row or column. Then there is for each ν only one P and for each Q there is at most one tuple of admissible semi-standard tableaux with shapes determined by Q and weights determined by P. The Q which have such a tuple are the semi-standard tableaux of weight ν if λ is a row and the row semi-standard tableaux of weight ν if λ is a column. Similar remarks apply to the case that μ consists of one row or column. The last two cases extend to prime characteristic, see Theorem 2 and Remark 3.1.
2. If we combine Theorem 4 with Theorem 1 we obtain, for k of characteristic 0, finite spanning sets for all spaces of highest weight vectors in $k[\mathcal{N}_{n,m}]$ and, taking $m = n - 1$, in $k[\mathcal{N}_n]$. Note that pulling the $u_{\nu, P, Q, \alpha}$ back just amounts to interpreting $x(Q(a))_{ij}$ as the (i, j)-th entry of the $Q(a)$-th matrix power and replacing $r - \pi(a) + 1$ by $n - \pi(a) + 1$. In particular, these pulled-back functions don't depend of the choice of r and s. In case of $\mathcal{N}_{n,m}$ the spanning sets are bases in all degrees for $n \geq (m + 1) \max(r, s)$. In case of $\mathcal{N}_n$ one can only say that in a fixed degree d the spanning sets will be bases if $n \geq l(\lambda) + l(\mu) + d - t$, where $t = |\lambda| = |\mu|$. This follows from Remark 2.1. Donin claimed in [7] and [8] that the spanning sets obtained above are always bases, but this is easily seen to be incorrect. Finding explicit homogeneous bases for all the spaces $k[\mathcal{N}]_\chi^{U_n}$

(or more generally $k[\overline{\mathcal{O}}_\eta]^{U_n}_\chi$) is still an open problem. If one tries to find them as subsets of the above spanning sets this is combinatorially already a challenging problem. In the case of the GL_n-modules $V^{\otimes r} \otimes (V^*)^{\otimes s}$, $V = k^n$, there is a similar problem of finding bases for the vector spaces $(V^{\otimes r} \otimes (V^*)^{\otimes s})^{U_n}_{[\lambda,\mu]}$. In [2] this was done for $n \geq l(\lambda) + l(\mu) + r - |\lambda|$. In this case there is at least a good candidate indexing set for arbitrary n: the up-down staircase tableaux of [21].

3. Note that in Theorem 4 we can choose each α the unique representative such that for all i α_i is admissible, i.e. a "picture" in the sense of [16] and [25].

4. The corollary to Theorem 4 proves a weaker version of the "conjecture" in [23, Sect. 4]: in the notation there, with $\chi = [\lambda, \mu]$, the elements

$$\vartheta\big(\psi_t((\tau, \mathrm{id}) \cdot E_\chi) \cdot s_{i_1} \otimes \cdots \otimes s_{i_t}\big),$$

$2 \leq i_1, \ldots, i_t \leq n$, $\tau \in \mathrm{Sym}_t$ generate the $k[\mathfrak{gl}_n]^{\mathrm{GL}_n}$-module $k[\mathfrak{gl}_n]^{U_n}_\chi$. This follows by pulling the spanning set of the corollary back to the nilpotent cone taking $m = n - 1$, using the fact that $(X^l)_{ij} = \pm(\partial_{ji}s_{l+1})(X)$ for all $X \in \mathcal{N}_n$, see [24, Cor to Thm 1], and applying Lemma 1. Of course one can also use the $\{\mathrm{id}\} \times \mathrm{Sym}_t$-conjugates of E_χ. Just take $\tau = \mathrm{id}$ and ρ in the proof of the corollary. The original conjecture is false, see [24, Rem 2.5].

3.3. Several matrices.

In this final section we look at highest weight vectors in the coordinate ring of the space of several matrices Mat^l_n under the diagonal conjugation action of GL_n. In order to be able to apply the graded Nakayama Lemma we need to work with the "null-scheme" rather then the null-cone. We will denote an l-tuple of $n \times n$-matrices $(X_1, \ldots, X_l)$ by $\underline{X}$.

We recall some results from [5, Sect. 4]. For i an integer ≥ 0 let $\mathcal{X}_i$ be the set of sequences of length $\leq i$ with entries in $\{1, \ldots, l\}$ and let $\mathcal{X}'_i$ be $\mathcal{X}_i$ with the empty sequence omitted. For $\eta \in \mathcal{X}_i$ of length $j \leq i$ define $f_\eta : \mathrm{Mat}^l_n \to \mathrm{Mat}_n$ by $f_\eta(\underline{X}) = X_{\eta_1} \cdots X_{\eta_j}$. By the Razmyslov-Procesi Theorem the algebra $k[\mathrm{Mat}_{rn} \times \mathrm{Mat}_{ns} \times \mathrm{Mat}^l_n]^{\mathrm{GL}_n}$ is generated by the functions $(A, B, \underline{X}) \mapsto \mathrm{tr}(f_\eta(\underline{X}))$ and $(A, B, \underline{X}) \mapsto (Af_\xi(\underline{X})B)_{ij}$, $\eta \in \mathcal{X}'_{n^2}$, $\xi \in \mathcal{X}_{n^2-1}$, $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, s\}$. Now let $\mathcal{M}_n$ be the closed subscheme of Mat^l_n corresponding to the ideal of $k[\mathrm{Mat}^l_n]$ generated by the functions $\underline{X} \mapsto \mathrm{tr}(f_\eta(\underline{X}))$. Then it follows from the above that for $m = |\mathcal{X}'_{n^2-1}|$ the restriction of the morphism

$$\psi_{r,s,n,l} : (A, B, \underline{X}) \mapsto (Af_\xi(\underline{X})B)_{\xi \in \mathcal{X}'_{n^2-1}} : Y_{r,s,n} \times \mathrm{Mat}^l_n \to \mathrm{Mat}^m_{rs}$$

to $Y_{r,s,n} \times \mathcal{M}_n$ is a GL_n-quotient morphism onto its scheme-theoretic image $\mathcal{W}_{n,l}$. Note that we omitted the empty sequence from $\mathcal{X}_{n^2-1}$, since we passed to $Y_{r,s,n}$, the variety of pairs of matrices $(A, B) \in \mathrm{Mat}_{rn} \times \mathrm{Mat}_{ns}$ with $AB = 0$.

Analogous to the case of one matrix we will identify Mat^l_n with the closed subvariety $\{(E_r, F_s)\} \times \mathrm{Mat}^l_n$ of $Y_{r,s,n} \times \mathrm{Mat}^l_n$ and denote the restriction of $\psi_{r,s,n,l}$ to Mat^l_n again by $\psi_{r,s,n,l}$. Then the union of the GL_n-conjugates of $\mathrm{Mat}^l_n = \{(E_r, F_s)\} \times \mathrm{Mat}^l_n$ is $\mathcal{O} \times \mathrm{Mat}^l_n$, where $\mathcal{O}$ consists of the pairs $(A, B) \in Y_{r,s,n}$ with $\mathrm{rk}(A) = r$ and $\mathrm{rk}(B) = s$. The same holds with Mat^l_n replaced by $\mathcal{M}_n$. It follows that the comorphism of $\psi_{r,s,n,l} : \mathcal{M}_n \to \mathcal{W}_{n,l}$ is injective, since the natural map $k[Y_{r,s,n} \times \mathcal{M}_n] \to k[\mathcal{O} \times \mathcal{M}_n]$ is injective. Furthermore, the analogue

of the identity for $\varphi_{r,s,n,m}$ at the beginning of the proof of Theorem 1 holds for $\psi_{r,s,n,l}$. Finally we apply the graded Nakayama Lemma to the $k[\mathrm{Mat}_n^l]^{\mathrm{GL}_n}$-module $k[\mathrm{Mat}_n^l]_\chi^{U_n}$ and we obtain

Theorem 5. *Let $\chi = [\lambda, \mu]$ be a dominant weight with coordinate sum zero and put $m = |\mathcal{X}'_{n^2-1}|$. Then the pull-back along $\psi_{r,s,n,l} : \mathrm{Mat}_n^l \to \mathrm{Mat}_{rs}^m$ of the spanning set of $k[\mathrm{Mat}_{rs}^m]_{(-\mu^{\mathrm{rev}},\lambda)}^{U_r \times U_s}$ from Theorem 4 or its corollary is a spanning set of the $k[\mathrm{Mat}_n^l]^{\mathrm{GL}_n}$-module $k[\mathrm{Mat}_n^l]_\chi^{U_n}$.*

Remarks 7. 1. Of course m above is huge, but if we are only interested in homogeneous highest weight vectors of degree d say, then we can take $m = |\mathcal{X}'_d|$ above and combine the resulting elements with homogeneous elements of $k[\mathrm{Mat}_n^l]^{\mathrm{GL}_n}$ to obtain a spanning set for the vector space of homogeneous highest weight vectors of weight χ and degree d.

2. Much of Section 3.3 generalises to prime characteristic, but it is not clear how to prove the analogue of Proposition 2 for several matrices.

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